Charles Baird
Edinburgh College
3rd Nov. 1846

A

GEOMETRICAL TREATISE

ON THE

CONIC SECTIONS;

WITH AN APPENDIX,

CONTAINING

FORMULÆ FOR THEIR QUADRATURE, &c.

BY

WILLIAM WALLACE, A.M., F.R.S., EDIN.

PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF EDINBURGH;
MEMBER OF THE ROYAL ASTRONOMICAL SOCIETY, AND
OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY, &c.

EDINBURGH:
ADAM AND CHARLES BLACK;
LONGMAN, REES, ORME, BROWN, GREEN, AND LONGMAN,
LONDON.

M.DCCC.XXXVII.

EDINBURGH:
Printed by THOMAS ALLAN & Co.
265 High Street.

PREFACE.

THE Treatise here offered to the public was composed for the ENCYCLOPÆDIA BRITANNICA, and has already appeared in the seventh edition of that national work. The Author has been induced to republish it in its present form by the hope that it may be found useful as a text-book, and afford the student of the mathematics some facilities in the acquisition of an interesting branch of the ancient geometry.

In treating of a subject which has passed through the hands of so many distinguished mathematicians, and which, on account of its numerous applications in astronomy and natural philosophy, has been considered under every point of view, the Author could not hope to add much to the stock of truths already known. His object, indeed, has not been so much to search after new propositions, or to point out relations heretofore unnoticed, as to place the subject be-

fore the reader in the clearest possible light, and to demonstrate the principal properties of the Conic Sections in a more concise, simple, and elementary manner than has yet been done.

With this view, perspicuity and symmetry have been particularly studied, both in the arrangement of the materials, and the demonstration of the particular propositions. Only the most elementary propositions of geometry have been made use of; and while each of the three sections has been treated as a distinct curve, a general view has been taken of the subject, and their analogous properties deduced from their respective definitions in an uniform manner, by the same constructions, and, in many instances, in the same words. It might have been easy, in such cases, to have included the three curves in the same general enunciation ; but the method which has been followed has the advantage of placing in each case a distinct object before the mind, at the same time that the connection and mutual relations of all the curves is rendered obvious by the comparison of those propositions in which their analogous properties are demonstrated.

In conformity with the same views, the curves have been considered as generated by the motion of a point

on a plane, and without any reference to the cone. In so far as facility of demonstration is concerned, it is perhaps of little consequence which of their characteristic properties is taken for the definition; that which has been adopted (and which was first employed by LAHIRE in his *Nouveaux Elemens des Sections Coniques*, Paris, 1679) appears to afford at least the simplest view of their mechanical description.

The Work consists of *Four Parts*, besides the *Appendix*. The first three parts contain the demonstration of the principal properties of each curve, considered separately and independently. The fourth part exhibits the origin of the curves from the intersection of a cone with a plane, according to the view taken of them by the ancient geometers, and from which, indeed, they derive their name of *Conic* Sections. It also embraces a subject of considerable importance in the theory of the curves, namely, the comparison of their curvature at each point, with that of a given circle; and it concludes with the demonstration of those properties of their areas which can be deduced without the aid of the higher geometry.

The Appendix is of a miscellaneous character. The first proposition, derived from a general property of the Conic Sections demonstrated by NEWTON, gives

an expeditious method of finding points in a parabola by the intersection of straight lines. Propositions 2, 3, 4, indicate how parabolas may be described which shall touch straight lines given in position. Propositions 5 and 6 unfold a very remarkable property of the ellipse, and show that it belongs to the class of curves called epicycloids, or hypocycloids, being generated by the motion of a point in the plane of a circle, which revolves on the interior circumference of another circle. This property has suggested the elegant instrument for the organic description of the ellipse described in the Scholium to Prop. 8. The equations of rectangular co-ordinates, and the varied expressions for the polar equations, are added for the purpose of facilitating the application of analysis to the investigation of the higher properties of the curves, and to astronomy. The series which follow for the areas of the circle and equilateral hyperbola, and also the remarkable properties of their circumscribed polygons, are in substance taken from a paper presented by the Author to the Royal Society of Edinburgh, and published in volume sixth of its Transactions

It was the intention of the Author to have considerably enlarged this part of the Work, and to have given various other series for the quadrature aud rec-

tification of the Conic Sections; but the state of his health having prevented him from accomplishing his wishes, and indeed delayed the publication of the Work much beyond the time he had hoped it would appear, he is induced to allow it to go forth in its present state; and if it shall be found to render the subject more accessible to the generality of students, or to promote a taste for that elegant species of geometrical investigation which was so successfully cultivated by the great masters of antiquity, and which, while it affords the best discipline for the minds of youth, furnishes also the securest foundation for the superstructure of the modern mathematics, he will not regret the time and labour which have been expended in its composition.

College of Edinburgh,
December 24, 1836.

INTRODUCTION.

THE mathematicians of antiquity regarded the straight line and the circle as the most simple of all geometrical lines; and the celebrated geometer Euclid has employed no other in his well-known *Elements*. By these alone the ancients resolved a great number of problems, of which the more simple are contained in Euclid's *Elements;* but many of higher difficulty were resolved in his other writings, and in treatises of Archimedes and Apollonius, which have only in part reached our times. There is, however, from the very nature of geometrical science, a limit to the applicability of the straight line and circle. Some problems admit of only one solution: these can be resolved by the intersection of two straight lines. Others again admit of two solutions, and such require lines which intersect each other in two points; therefore they may be resolved by the straight line and circle, or two circles.

If, however, a problem be of such a nature as in its most general form to admit of three solutions, it must necessarily be determined by the intersection of two lines which intersect each other in three points; it therefore cannot be resolved by the straight line and circle alone. Now the ancients had actually proposed to themselves such problems, and in this way it may be supposed they had

discovered the necessity of introducing other lines, in addition to the straight line and circle, into their geometry.

The interest which mankind take in mathematical speculations is but little in comparison to that which is excited by works of poetry, oratory, or history; hence it has happened that ancient treatises on these subjects have had a better chance of descending to our times. It is not, therefore, wonderful that none of the works of the more early Greek geometers have reached us, and that we have no work of great antiquity professedly written on the subject of the Conic Sections. Our curiosity must therefore rest satisfied with the knowledge of a few incidental notices, and facts relating to them, gleaned from different authors.

The discovery of the Conic Sections seems to have originated in the school of Plato, in which geometry was highly esteemed and much cultivated. It is probable that the followers of that philosopher were led to the discovery of these curves, and to the investigation of many of their properties, in seeking to resolve the two famous problems of the duplication of the cube, and the trisection of an angle, for which the artifices of the ordinary or plane geometry were insufficient. Two solutions of the former problem, by the help of the Conic Sections, are preserved by Eutocius,* and are attributed by him to Menæchmus, the scholar of Eudoxus, who lived not much posterior to the time of Plato: and this circumstance, added to a few words in an epigram of Eratosthenes,† has been thought sufficient authority, by some authors, to ascribe the honour of the discovery of the Conic Sections to Menæchmus. We may at least infer that, at this epoch, geometers had made some progress in developing the properties of these curves.

* In Arch. lib. ii. *De Sph. et Cyl.*

† Ibid.

The writings of Archimedes that have reached us explicitly show that the geometers before his time had advanced a great length in investigating the properties of the Conic Sections. This author expressly mentions many principal propositions to have been demonstrated by preceding writers; and he often refers to properties of the Conic Sections, as truths commonly divulged and known to mathematicians. His own discoveries in this branch of science are worthy of the most profound and inventive genius of antiquity. In the quadrature of the parabola he gave the first and the most remarkable instance that has yet been discovered of the exact equality of a curvilinear to a rectilinear space. He determined the proportion of elliptic spaces to corresponding spaces in the circle; and he invented many propositions respecting the mensuration of the solids formed by the revolution of the conic sections about their axes.

It is chiefly from the writings of Apollonius of Perga, a town in Pamphylia, on the subject of the Conic Sections, that we know how far the ancient mathematicians carried their speculations concerning these curves. Apollonius flourished under Ptolemy Philopater, about forty years later than Archimedes. He formed his taste for geometry, and acquired that superior skill in the science to which he is indebted for his fame, in the school of Alexandria, under the successors of Euclid. Besides his great work on the Conic Sections, he was the author of many smaller treatises relating chiefly to the geometrical analysis, the originals of which have all perished, and are only known to modern mathematicians by the account given of them by Pappus of Alexandria, in the seventh book of his *Mathematical Collections*.

The work of Apollonius on the Conic Sections, written in eight books, was held in such high estimation by the ancients, as to procure for him the name of the *Great Geometer*. Only the first four books of this treatise have come down to us in the original Greek. It is the purpose of these, as we are informed in the prefatory epistle to Eudemus, to deliver the elements of the science; and in this part of his labour the author claims no further merit than that of having collected, amplified, and reduced to order, the discoveries of preceding mathematicians. One improvement introduced by Apollonius is too remarkable to be passed over without notice. The geometers who preceded him derived each curve from a *right cone*, which they conceived to be cut by a plane perpendicular to its slant side. It will readily be perceived, from what is shown in the first section of the fourth part of the following treatise, that the section would be a parabola when the vertical angle of the cone was a right angle, an ellipse when it was acute, and an hyperbola when it was obtuse. Thus each curve was derived from a different sort of cone. Apollonius was the first to show that all the curves may be produced from any sort of cone, whether right or oblique, according to the different inclinations of the cutting plane. This fact is one remarkable instance of the adherence of the mind to its first conceptions, and of the slowness and difficulty with which it generalizes.

The original of the last four books of the treatise of Apollonius is lost; and it is not easy to ascertain in what age it disappeared. In the year 1658 Borelli discovered at Florence an Arabic manuscript, entitled *Apollonii Pergæi Conicorum Libri Octo.* By the liberality of the Duke of Tuscany, he was permitted to carry the manuscript to

Rome, and, with the aid of an Arabic scholar, Abraham Ecchellensis, he published in 1661 a Latin translation of it. The manuscript, although from its title it was expected to be a complete translation of all the eight books, yet was found to contain only the first seven books: and it is remarkable, that another manuscript, brought from the East by Golius, the learned professor of Leyden, in 1664, as well as a third, of which Ravius published a translation in 1669, have the same defect. All the three manuscripts agreeing in the want of the eighth book, we may now consider that part of the work of Apollonius as irrecoverably lost. Fortunately, in the *Collectiones Mathematicæ* of Pappus, in whose time the entire treatise of Apollonius was extant, there is preserved some account of the subjects treated in each book, and all the *Lemmata* required in the investigations of the propositions they contained. Dr Halley, who in 1710 gave a correct edition of the Conics of Apollonius, guided in his researches by the lights derived from Pappus, has restored the eighth book with so much ability as to leave little reason to regret the loss of the original.

The last four books of the Conics of Apollonius, containing the higher or more recondite parts of the science, are generally supposed to be the fruit of the author's own researches; and they do much honour to the geometrical skill and invention of the great geometer. Even in our time the whole treatise must be regarded as a very extensive, if not a complete work on the Conic Sections. Modern mathematicians make important applications of these curves, with which the ancients were unacquainted; and they have been thus led to consider the subject in particular points of view, suited to their purposes; but

they have made few discoveries of which there are not some traces to be found in the work of the illustrious ancient.

The geometers who followed Apollonius seem to have contented themselves with the humble task of commenting on his treatise, and of rendering it of more easy access to the bulk of mathematicians. Till about the middle of the 16th century, the history of this branch of mathematical science presents nothing remarkable. The study of it was then revived; and since that time this part of the mathematics has been more cultivated, or has been illustrated by a greater variety of ingenious writings.

Among the ancients the study of the Conic Sections was a subject of pure intellectual speculation. The applications of the properties of these curves in natural philosophy have, in modern times, given to this part of the mathematics a degree of importance that it did not formerly possess. That which, in former times, might be considered as interesting only to the learned theorist and profound mathematician, is now a necessary attainment to him who would not be ignorant of those discoveries in nature that do the greatest honour to the present age.

It is curious to remark, in the progress of discovery, the connexion that subsists between the different branches of human knowledge; and it excites admiration to reflect, that the astronomical discoveries of Kepler, and the sublime theory of Newton, depend on the seemingly barren speculations of Greek geometers concerning the sections of the cone.

Apollonius, and all the writers on Conic Sections before Dr Wallis, derived the elementary properties of the curves from the nature of the cone. In the second part of his

treatise *De Sectionibus Conicis*, published in 1665, Dr Wallis laid aside the consideration of the cone, deriving the properties of the curves from a description *in plano*. Since his time authors have been much divided as to the best method of defining those curves, and demonstrating their elementary properties; many of them preferring that of the ancient geometers, while others, and some of great note, have followed his example.

In support of the innovation made by Dr Wallis, it is urged, that in the ancient manner of treating the Conic Sections, students are perplexed and discouraged by the previous matter to be learnt respecting the generation and properties of the cone; and that they find it no easy task to conceive distinctly, and to understand, diagrams which represent lines drawn in different planes; all these difficulties are avoided by defining the curves *in plano* from some one of their essential properties. It is not our intention particularly to discuss this point; and we have only to add, that in the following treatise we have chosen to deduce the properties of the Conic Sections from their description *in plano*, as better adapted to the nature of an elementary treatise.

A geometrical treatise on the Conic Sections must necessarily be founded upon the elements of geometry. As Euclid's *Elements of Geometry* are generally studied, and in every one's hands, we have chosen to refer to it in the demonstrations. The edition referred to is that published by the late Professor Playfair of Edinburgh.

The references are to be thus understood: (20, 1, E.) means the twentieth proposition of the first book of Euclid's *Elements;* (2 Cor. 20, 6, E.) means the second corollary to the twentieth proposition of the sixth book of the same

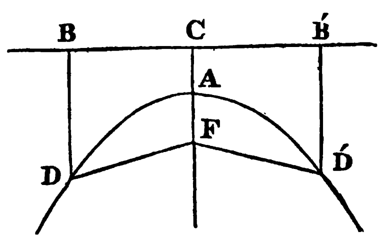
work; and so of others. Again, (7) means the seventh proposition of that Part of the following Treatise in which such reference happens to occur; (Cor. 1) means the corollary to the first proposition; (2 Cor. 3) means the second corollary to the third proposition, &c.—such references being all made to the propositions in the division of the Treatise in which they are found.

CONIC SECTIONS.

PART I.

OF THE PARABOLA.

DEFINITIONS.



I. If a straight line BC, and a point without it F, be given in position in a plane, and a point D be supposed to move in such a manner that DF, its distance from the given point, is equal to DB, its distance from the given line; the point D will describe a line DAD′, called a *Parabola.*

II. The straight line BC, which is given in position, is called the *Directrix of the Parabola.*

III. The given point F is called the *Focus.*

IV. A straight line perpendicular to the directrix, terminated at one extremity by the parabola, and produced indefinitely within it, is called a *Diameter.*

V. The point in which a diameter meets the parabola is called its *Vertex.*

VI. The diameter which passes through the focus is called the *Axis of the Parabola ;* and the vertex of the axis is called the *Principal Vertex.*

COROLLARY. A perpendicular drawn from the focus to the directrix is bisected at the vertex of the axis.

VII. A straight line terminated both ways by the parabola, and bisected by a diameter, is called an *Ordinate to that Diameter.*

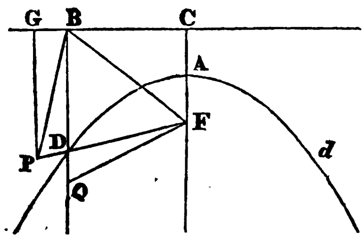
VIII. The segment of a diameter between its vertex and an ordinate is called an *Absciss.*

IX. A straight line quadruple the distance between the vertex of a diameter and the directrix, is called the *Parameter,* also the *Latus Rectum of that Diameter.*

X. A straight line meeting the parabola only in one point, and which everywhere else falls without it, is said to *touch* the parabola at that point, and is called a *Tangent to the Parabola.*

PROPOSITION I.

The distance of any point without the parabola from the focus is greater than its distance from the directrix; and the distance of any point within the parabola from the focus is less than its distance from the directrix.



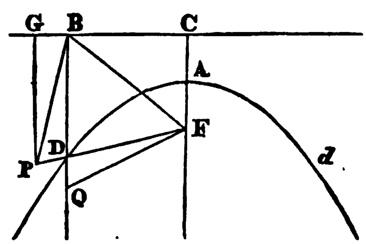
Let DA*d* be a parabola, of which F is the focus, GC the directrix, and P a point without the curve, that is, on the same side of the curve with the directrix; PF, a line drawn to the focus, will be greater than PG, a perpendicular to the directrix. For, as PF must necessarily cut the curve, let D be the point of intersection; draw DB perpendicular to the directrix, and join PB. Because D is a point in the parabola, DB = DF (Definition 1), therefore PF = PD + DB; but PD + DB is greater than PB (20, 1, E.), and therefore still greater than PG (19, 1, E.), therefore PF is greater than PG.

Again, let Q be a point within the parabola; QF, a line drawn to the focus, is less than QB, a perpendicular to the directrix. The perpendicular QB necessarily cuts the curve; let D be the point of intersection; join DF. Then DF = DB (Def. 1), and QD + DF = QB; but QF is less than DQ + DF, therefore QF is less than QB.

Cor. A point is without or within the parabola, according as its distance from the focus is greater or less than its distance from the directrix.

PROPOSITION II.

Every straight line perpendicular to the directrix meets the parabola, and every diameter falls wholly within it.

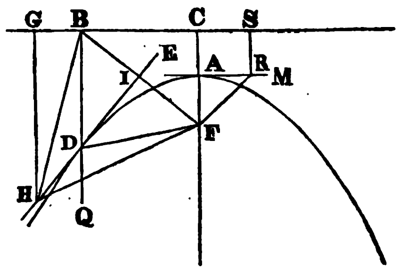


Let the straight line BQ be perpendicular to the directrix at B; BQ shall meet the parabola. Draw BF to the focus, and make the angle BFP equal to FBQ; then, because QBC is a right angle, QBF and PFB are each less than a right angle, therefore QB and PF intersect each other; let D be the point of intersection, then DB = DF (5, 1, E.); therefore D is a point in the parabola. Again, the diameter DQ falls wholly within the parabola; for take Q, any point in the diameter, and draw FQ to the focus, then QB or QD + DF is greater than QF; therefore Q is within the parabola (Cor. 1).

Cor. The parabola continually recedes from the axis, and a point may be found in the curve that shall be at a greater distance from the axis than any assigned line.

PROPOSITION III.

The straight line which bisects the angle contained by two straight lines drawn from any point in the parabola, the one to the focus, and the other perpendicular to the directrix, is a tangent to the curve in that point.



Let D be any point in the curve; let DF be drawn to the focus, and DB perpendicular to the directrix; the straight line which bisects the angle FDB is a tangent to the curve. Join BF meeting DE in I, take H any other point in DE, join HF, HB, and draw HG perpendicular to the directrix. Because DF = DB, and DI is common to the triangles DFI, DBI, and the angles FDI, BDI, are equal, these triangles are equal, and FI = IB; and hence FH = HB (4, 1, E.): but HB is greater than HG (19, 1, E.), therefore the distance of the point H from the focus is greater than its distance from the directrix; hence that point is without the parabola (Cor. 1), and therefore HDI is a tangent to the curve at D (Def. 10).

Cor. 1. A perpendicular to the axis at its vertex is a tangent to the curve. Let AM be perpendicular to the axis at the vertex A, then RS, the distance of any point

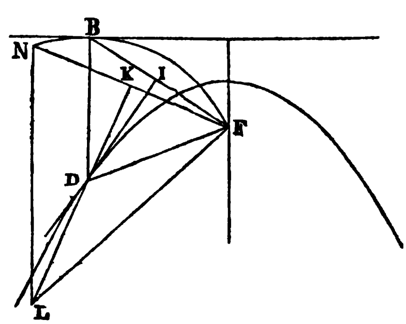
in AM from the directrix, is equal to CA, that is to AF, and therefore is less than RF, the distance of the same point from the focus.

COR. 2. A straight line drawn from the focus of a parabola perpendicular to a tangent, and produced to meet the directrix, is bisected by the tangent. For it has been shown that FB, which is perpendicular to the tangent DI, is bisected at I.

COR. 3. A tangent to the parabola makes equal angles with the diameter which passes through the point of contact, and a straight line drawn from that point to the focus. For BD being produced to Q, DQ is a diameter, and the angle HDQ is equal to BDE, that is, to EDF.

COR. 4. The axis is the only diameter which is perpendicular to a tangent at its vertex. For the angle HDQ, or BDE, is the half of BDF, and therefore less than a right angle, except when BD and DF lie in a straight line, which happens when D falls at the vertex.

COR. 5. There cannot be more than one tangent to the parabola at the same point.



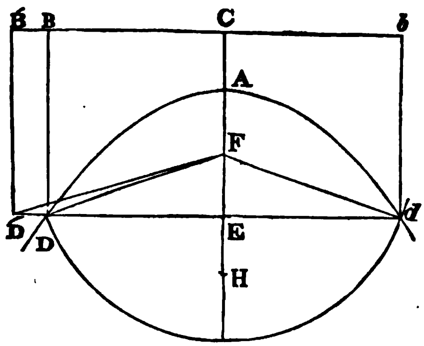
For let any other line DK, except a diameter, be drawn through D; draw FK perpendicular to DK; on D for a

centre, with a radius equal to DB or DF, describe a circle, cutting FK in N; draw NL parallel to the axis, meeting DK in L, and join FL. Then FK = KN (3, 3, E.), and therefore FL = LN. Now BD being perpendicular to the directrix, the circle FBN touches the directrix at B (16, 3, E.); and hence N, any other point in the circumference, is without the directrix, and on the same side of it as the parabola; therefore the point L is nearer to the focus than to the directrix, and consequently is within the parabola.

SCHOLIUM. From the property of tangents to the parabola demonstrated in Cor. 3, the point F takes the name of the *Focus*. For rays of light proceeding parallel to the axis of a parabola, and falling upon a polished surface whose figure is that produced by the revolution of the parabola about its axis, are reflected to the focus.

PROPOSITION IV. PROBLEM.

To find any number of points in a parabola, having given the focus and axis.



Let F be the focus, AH the axis, and A the vertex. Suppose the problem resolved, and that D is a point in the parabola. In FA produced take AC equal to AF, and through C draw the directrix BC*b*: draw DF to the focus, DE perpendicular to the axis, and DB perpendicular to the directrix: Take AH equal to FD.

Because AH is equal to DF, and DF is greater than AF (Cor. Prop. 2), therefore AH is greater than AF, and H is always in AF produced.

Now CE is equal to AH, for each is equal to DF; therefore, taking from these the equals AC, AF, we have AE = FH.

Construction. In AF produced take any point H, and take AE equal to FH. Through E draw DE*d* perpendicular to the axis, and with F as a centre, at the distance AH, describe a circle which will cut the perpendi-

cular in D and d: these are points in the parabola. For AE = FH, therefore CE = AH, and DB = DF, therefore D is in the parabola, and in the same way it appears that d is in the parabola.

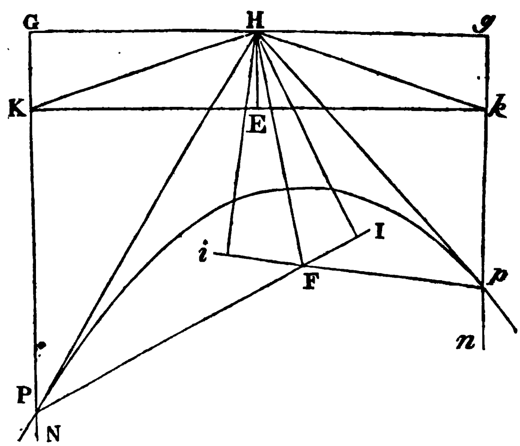
Cor. 1. Any perpendicular to the axis meets the parabola in two points, and in no more, and the straight line between the points is bisected by the axis. For if the perpendicular could meet the curve in another point D′, then FD′ being joined, we would have FD′ equal to D′B′, that is, to DB or to FD, which is impossible (19, 1, E.).

Cor. 2. Every chord Dd, in a parabola, perpendicular to the axis, is bisected by the axis, and therefore is an ordinate to it. For the chord in the parabola is also a chord in a circle, the centre of which is in the axis of the parabola.

Scholium. From this proposition it appears that the parabola is composed of two branches, which recede continually from the directrix and from the focus, also from the axis (Cor. Prop. 2). And it appears that the indefinite spaces between the curve and axis on each side are exactly alike, so that if the whole space comprehended within the parabola were divided into two portions by cutting it through the axis, and one of them were turned over upon the other, they would entirely coincide.

PROPOSITION V.

If a straight line be drawn from the focus of a parabola to the intersection of two tangents to the curve; it makes equal angles with straight lines drawn from the focus to the points of contact.



Let HP, Hp, tangents to a parabola at P and p, intersect each other at H; draw PF, pF, HF, to F the focus; the line HF makes equal angles with FP, Fp.

Draw PK, pk perpendicular to the directrix, and join HK, Hk. The triangles HPK, HPF have PK = PF, PH common to both, and the angles KPH, FPH equal (3), therefore they are in every way equal (4, 1, E.), and have HK = HF, and the angle HKP equal to HFP. In the same way it may be shown, that the triangles Hpk, HpF, are in every way equal, and therefore Hk = HF, and the angle HKp is equal to HFp: But HK being equal to Hk,

for each has been proved equal to HF, the angles HKk, HkK are equal (5, 1, E.), and adding to these the right angles PKk, pkK, the angles HKP, Hkp are equal; but these have been proved equal to HFP, HFp; therefore these last are equal, and the line HF makes equal angles with FP, Fp.

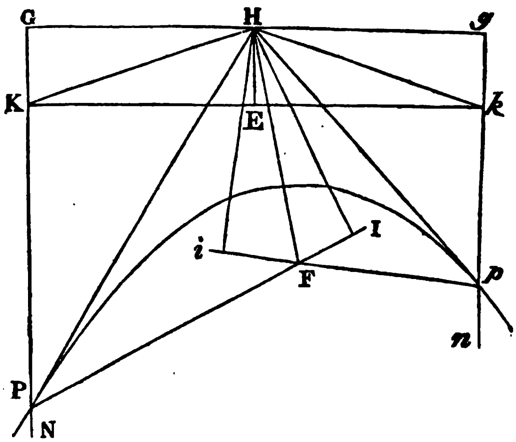
Cor. 1. Perpendiculars drawn from the intersection of two tangents, to lines drawn from the points of contact through the focus, are equal. For HI, Hi, being drawn perpendicular to PF, pF, the triangles HFI, HFi, are manifestly equal (26, 1, E.), and therefore HI = Hi.

Cor. 2. Perpendiculars from the intersection of two tangents to diameters passing through the points of contact are equal.

Draw GHg through H perpendicular to PK, pk, and because the triangles HPG, HPI have HP common to both, the angles at P equal, and the angles G and I right angles, the triangles are in every way equal (26, 1, E.), and hence HG is equal to HI. In like manner it is proved that Hg is equal to Hi; but HI is equal to Hi, therefore HG is equal to Hg.

PROPOSITION VI.

If a straight line be drawn from the intersection of two tangents to the focus, and another perpendicular to the directrix; these will make equal angles with the tangents.



Let F be the focus of a parabola, and Kk the directrix; and let straight lines HP, Hp, which intersect each other at H, touch the parabola at P and p; also let HF be drawn to the focus, and HE perpendicular to the directrix; the angles PHE, pHF are equal.

The same construction being made as in Prop. V.

In the triangles HEK, HEk, it may be shown, as in that proposition, that HK is equal to Hk, and therefore that the angle HKE is equal to the angle HkE (5, 1, E.). The angles HEK, HEk are also equal; therefore the angles KHE, kHE are equal (26, 1, E.). Now the angle KHE = KHP + PHE; but the triangles KHP, FHP are

in every way equal (as was shown in Prop. V.). Therefore KHP = FHP, and hence

$$KHE = FHP + PHE = FHE + 2PHE.$$

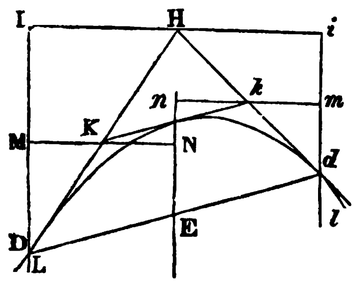
In the same way it appears that

$$kHE = FHp + pHE = FHE + 2FHp;$$

therefore $FHE + 2PHE = FHE + 2FHp$;
and hence $2PHE = 2pHF$ and $PHE = pHF$.

PROPOSITION VII.

If two tangents to a parabola be at the extremities of a chord, and a third tangent be parallel to the chord; the part of this tangent intercepted between the other two is bisected at the point of contact.



Let HD, Hd be tangents at the extremities of the chord Dd, and KPk a tangent parallel to Dd, meeting the other tangents in K and k; the line Kk is bisected at P, the point of contact.

From H, K, k, the intersections of the tangents, draw perpendiculars to the diameters passing through their points of contact, viz. HI, Hi, perpendicular to DL and dl; and KM, KN, perpendicular to DL and PE, and km, kn, perpendicular to dl and PE.

The triangles HDI, DKM, are manifestly equiangular, also the triangles dHi, dkm; therefore

$$\text{HD} : \text{DK} = \text{HI} : \text{KM} \ (4, 6, \text{E.}),$$

$$\text{and } \text{H}d : dk = \text{H}i : km.$$

But because Kk is parallel to Dd,

$$\text{HD} : \text{DK} = \text{H}d : dk \ (2, 6, \text{E.});$$

$$\text{therefore } \text{HI} : \text{KM} = \text{H}i : km.$$

Now HI = Hi (2 Cor. 5), therefore KM = km.

But $KM = KN$, and $Km = Kn$ (2 Cor. 5);

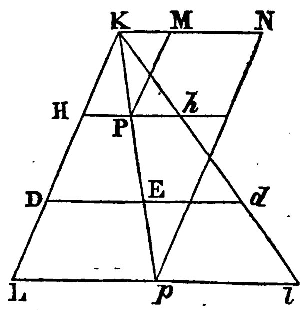
therefore $KN = Kn$.

And since $KN : kn = KP : kP$ (Cor. 6, E.),

therefore $KP = kP$.

Lemma.

Let KLl be a triangle, having its base Ll bisected at p, and let Hh, any straight line parallel to the base, and terminated by the sides, be bisected at P; then P, p, the points of bisection, and K, the vertex of the triangle, are in the same straight line; and that line bisects Dd, any other straight line parallel to the base.



Complete the parallelograms KHPM, $KLpN$. The triangles KHh, KLl being similar, and Hh, Ll similarly divided at P and p,

$$KH : KL = Hh : Ll = HP : Lp,$$

hence the parallelograms KHPM, $KLpN$ are similar. Now they have a common angle at K, therefore they are about the same diameter, that is, the points K, P, p, are in the same straight line (26, 6, E.).

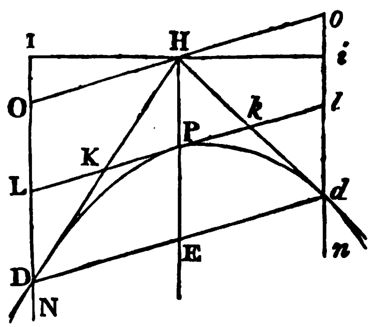
Next, let Dd meet Kp in E, then

$$HP : DE \; (= KP : KE) = Ph : Ed;$$

therefore DE is equal to Ed.

PROPOSITION VIII.

Any chord parallel to a tangent is bisected by the diameter which passes through the point of contact, or is an ordinate to that diameter.



The chord D*d*, which is parallel to the tangent KP*k*, is bisected at E by PE, the diameter that passes through the point of contact.

Let DH, *d*H be tangents, and DN, *dn* diameters at the extremities of the chord; let the tangent at P meet the other tangents in K and *k*, and the diameters in L and *l*, and through H draw OH*o* parallel to D*d*, and IH*i* perpendicular to the diameters DN, *dn*.

Because of the parallels L*l*, O*o*, and DO, *do*, the triangles DKL, DHO are similar, also the triangles *dkl*, *d*H*o*, and the triangles OHI, *o*H*i*, therefore

$$\text{DK} : \text{DH} = \text{KL} : \text{HO},$$

$$\text{and } dk : d\text{H} = kl : \text{H}o \text{ (4, 6, E.)};$$

But because D*d* is parallel to K*k*,

$$\text{DK} : \text{DH} = dk : d\text{H} \text{ (2, 6, E.)},$$

$$\text{therefore KL} : \text{HO} = kl : \text{H}o;$$

but HO : HI = H*o* : H*i*,

therefore, *ex. æq.* KL : HI = *kl* : H*i*.

Now HI = H*i* (2 Cor. 5), therefore KL = *kl*; but KP = *k*P (7), therefore PL = P*l*, and ED = E*d* (34, 1, E.)

Cor. 1. Straight lines which touch a parabola at the extremities of an ordinate to a diameter intersect each other in that diameter; for K*k* and D*d* being bisected at P and E, the points H, P, E lie in a straight line. (Lemma.)

Cor. 2. Every ordinate to a diameter is parallel to a tangent at its vertex: For if it be not, let a tangent be drawn parallel to the ordinate; then the diameter which passes through the point of contact would bisect the ordinate, and thus the same line would be bisected in two different points, which is impossible.

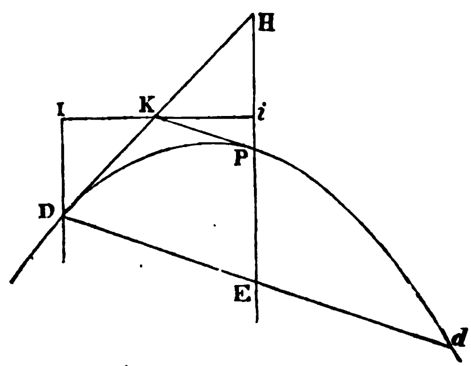
Cor. 3. All ordinates to the same diameter are parallel to each other.

Cor. 4. A straight line that bisects two parallel chords, and terminates in the curve, is a diameter.

Cor. 5. The axis is perpendicular to its ordinates, and every other diameter cuts its ordinates obliquely.

PROPOSITION IX.

If a tangent at any point in a parabola meet a diameter, and from the point of contact an ordinate be drawn to that diameter; the segment of the diameter between the vertex and tangent is equal to the segment between the vertex and the ordinate.



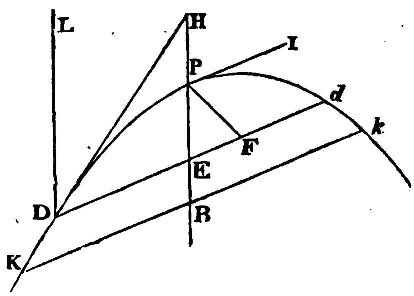
Let DH, a tangent to the curve at D, meet the diameter EP in H, and let DE*d* be an ordinate to that diameter: the segment HE is bisected in P.

Draw PK, a tangent at P, meeting the tangent DH in K, and draw IK*i*, perpendicular to the diameter PE at *i*, and meeting a diameter drawn through D at I.

The triangles DKI, HK*i* are equiangular (29, 1, E.), therefore IK : K*i* = DK : KH (4, 6, E.); and because in the triangle DHE, KP is parallel to the side DE, DK : KH = EP : PH, therefore IK : K*i* = EP : PH; but IK and K*i* are equal (2 Cor. 5), therefore EP and PH are equal.

PROPOSITION X.

If an ordinate to any diameter pass through the focus; the absciss is equal to one fourth of the parameter of that diameter, and the ordinate is equal to the whole parameter.

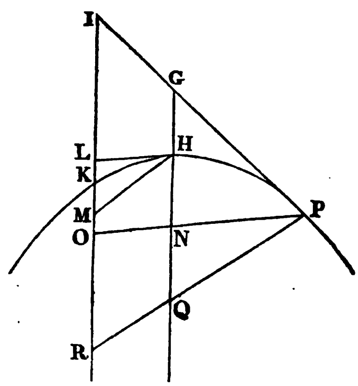


Let DE*d*, a straight line passing through the focus, be an ordinate to the diameter PE; the absciss PE is equal to one fourth of the parameter, and the ordinate D*d* is equal to the whole parameter of the diameter PE.

Let DH, PI be tangents at D and P; let DH meet the diameter in H; draw PF to the focus, and DL parallel to EP. The angles HPI, IPF, being equal (3), and PI parallel to EF (2 Cor. 8), the angles PEF, PFE, are also equal (29, 1, E.), and PE = PF = $\frac{1}{4}$ the parameter (Def. 9 and Def. 1). Again, the angle HDE is equal to LDH (3), and therefore equal to DHE; consequently ED is equal to EH, or to twice EP (9): therefore D*d* is equal to 4EP, or to 4PF, that is, to the parameter of the diameter.

PROPOSITION XI.

If any two diameters of a parabola be produced to meet a tangent to the curve; the segments of the diameters between their vertices and the tangent are to one another as the squares of the segments of the tangent intercepted between each diameter and the point of contact.



Let QH, RK, any two diameters, be produced to meet PI, a tangent to the curve at P, in the points G, I; then,

$$HG : KI = PG^2 : PI^2.$$

Let PN, a semi-ordinate to the diameter HQ, meet KR in O, and let PR, a semi-ordinate to the diameter KO, meet HN in Q; from H draw parallels to NO and QR, meeting KR in L and M; thus HL is a tangent to the curve, and HM a semi-ordinate to KR.

Now $KI = KR$, and $KL = KM$ (9);

therefore, by subtraction, $LI = MR = HQ$.

$$\text{But } LO = HN = HG \ (9);$$

therefore, by addition, $IO = GQ$.

The triangles PGN, PIO, are similar, as also PGQ, PIR,

$$\text{therefore } GN, \text{ or } 2GH : IO = PG : PI,$$

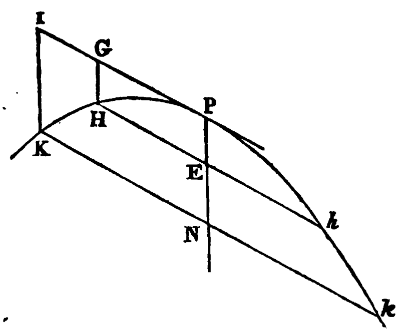
$$\text{and } GQ : IR, \text{ or } IO : 2IK = PG : PI.$$

Hence, taking the rectangles of the corresponding terms,

$$2GH \cdot IO : IO \cdot 2IK = PG^2 : PI^2,$$

$$\text{therefore } GH : IK = PG^2 : PI^2.$$

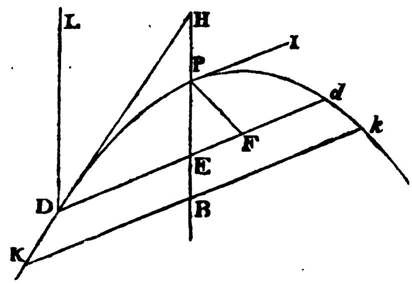
Cor. The squares of semi-ordinates, and of ordinates to any diameter, are to one another as their corresponding abscisses.



Let HEh, KNk be ordinates to the diameter PN; draw PG a tangent to the curve at the vertex of the diameter, and complete the parallelograms PEHG, PNKI; then PG, PI are equal to EH, NK, and GH, IK to PE, PN respectively; therefore $HE^2 : KN^2 = PE : PN$

PROPOSITION XII.

If an ordinate be drawn to any diameter of a parabola; the rectangle under the absciss and the parameter of the diameter is equal to the square of the semi-ordinate.



Let KBk be an ordinate to the diameter PB; the rectangle contained by PB and the parameter of the diameter is equal to the square of KB, the semi-ordinate.

Let DEd be that ordinate to the diameter which passes through the focus. The semi-ordinates DE, Ed are each half of the parameter, and the absciss EP is one fourth of the parameter (10);

$$\text{therefore } Dd : DE = DE : PE,$$

$$\text{and } Dd \cdot PE = DE^2 \text{ (16, 6, E.).}$$

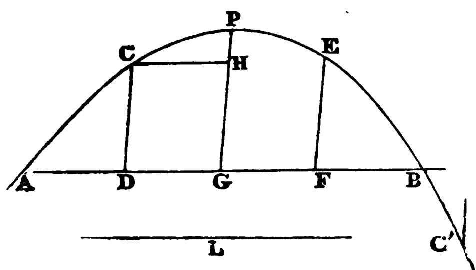
$$\text{But } Dd \cdot PE : Dd \cdot PB, \text{ or } PE : PB = DE^2 : KB^2 \text{ (Cor. 11),}$$

$$\text{therefore } Dd \cdot PB = KB^2.$$

SCHOLIUM. It was on account of the equality of the square of the semi-ordinate to a rectangle contained by the parameter of the diameter and the absciss, that Apollonius called the curve line to which the property belonged a *Parabola*.

PROPOSITION XIII.

If AB, *an ordinate to a diameter* PG, *be cut by any other diameter* CD *in* D; *the rectangle* AD · DB *contained by its segments is equal to the rectangle contained by* CD, *the segment of the other diameter between its vertex and the ordinate, and the parameter of the diameter* PG.



Draw CH, a semi-ordinate to the diameter PG, and let L be its parameter.

$$\text{Because } AG^2 = L \cdot PG \ (12),$$
$$\text{and } DG^2 = CH^2 = L \cdot PH,$$
$$\text{therefore } AG^2 - DG^2 = L\,(PG - PH),$$
$$\text{that is } (5, 2, \text{E.}),\ AD \cdot DB = L \cdot CD.$$

When the point D′ is in AB produced, the demonstration requires Prop. 6, instead of Prop. 5 of 2, E.

Cor. If a chord AB be cut by any two diameters CD, EF, the rectangles AD · DB, AF · FB, are to one another as CD, EF, the segments of the diameters between their vertices and the chord.

For since AD · DB = L · CD; and AF · FB = L · EF; AD · DB : AF · FB = L · CD : L · EF = CD : EF.

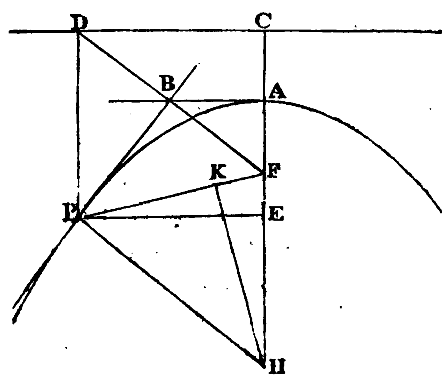
Definitions.

XI. The straight line PH, perpendicular to a tangent between the point of contact P and the axis AH, is called a *Normal.*

XII. The segment EH of the axis between the normal and PE, an ordinate to the axis drawn through P, is called a *Subnormal.*

PROPOSITION XIV.

A straight line drawn from the focus of a parabola, perpendicular to a tangent, is a mean proportional between the straight line drawn from the focus to the point of contact, and one fourth the parameter of the axis.



Let FB be a perpendicular from the focus upon the tangent PB, and FP a straight line drawn to the point of contact; let A be the principal vertex, and therefore FA equal to one fourth of the parameter of the axis; FB is a mean proportional between FP and FA.

Produce FB and FA to meet the directrix in D and C, and join AB. The lines FC, FD are bisected at A and B (2 Cor. 3), therefore (2, 6, E.) AB is parallel to CD, or perpendicular to CF, and consequently is a tangent to the curve at A (1 Cor. 3). Now BP is a tangent at P, therefore the angle AFB is equal to BFP (5); and since the angles FAB, FBP are right angles, the triangles FAB, FBP are equiangular; hence

$$FP : FB = FB : FA.$$

Cor. 1. The common intersection of a tangent, and a perpendicular from the focus to the tangent, is in a straight line touching the parabola at its vertex.

Cor. 2. If PH be drawn perpendicular to the tangent, meeting the axis in H, and HK be drawn perpendicular to PF; PK shall be equal to half the parameter of the axis. For the triangles HPK, FBP, are manifestly equiangular; therefore

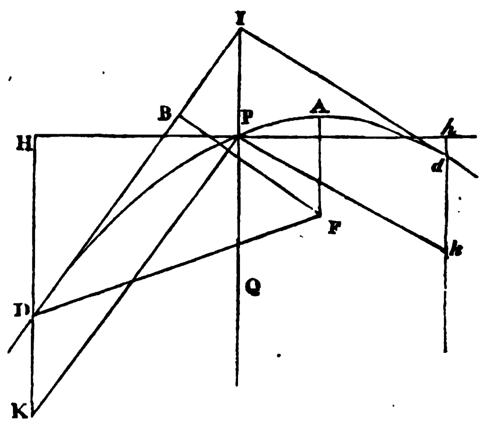
$$HP : PK = PF : FB = FB : FA = FD : FC.$$

But if PD be joined, the line PD is evidently perpendicular to the directrix (3), therefore the figure HPDF is a parallelogram, and HP = FD, therefore PK = FC = half the parameter of the axis.

Cor. 3. The subnormal HE, is equal to half the parameter of the axis. For the triangles PHE, DFC are in all respects equal; therefore HE = FC.

PROPOSITION XV.

If from a point in a parabola a perpendicular be drawn to any diameter, and also, from the same point, an ordinate to that diameter; the square of the perpendicular is equal to the rectangle contained by the absciss of the diameter and the parameter of the axis.



Let P be a point in a parabola, DK any diameter, PH a perpendicular, and PK a semi-ordinate to that diameter; the square of PH is equal to the rectangle contained by DK and the parameter of the axis.

Let F be the focus, and FA the segment of the axis between the focus and vertex, and therefore one fourth of the parameter; join FD, draw DB touching the parabola at D, and FB a perpendicular from the focus on the tangent. The triangles PKH, FDB are similar, for the angle FDB is equal to BDH (3 Cor. 3), that is, to PKH

(2 Cor. 8), and the angles at B and H are right angles, therefore their sides are proportionals.

and $KP^2 : PH^2 = DF^2 : FB^2$.

But since $DF : FB = FB : FA$ (14),

$DF^2 : FB^2 = DF : FA$ (2 Cor. 20, 6, E.);

therefore $KP^2 : PH^2 = DF : FA = 4DF{\cdot}DK : 4FA{\cdot}DK$.

Now $KP^2 = 4\,DF{\cdot}DK$ (12), therefore $PH^2 = 4\,FA{\cdot}DK$.

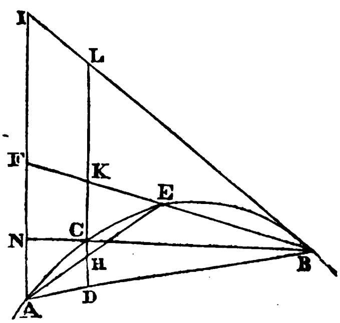
Cor. 1. Hence, if two diameters DK, *dk*, on opposite sides of a third PQ, be at equal distances from it, semi-ordinates PK, P*k* to the other two, drawn from P the vertex of the middle diameter, will cut off equal abscisses DK, *dk*. For the perpendiculars PH, P*h*, on the two extreme diameters, are equal.

Cor. 2. And if tangents DI, *d*I, be drawn at the vertices of the extreme diameters, they will intersect each other in the middle diameter QP produced. For the tangents being parallel to the ordinates (8), each will cut off from PQ a segment PI equal to the absciss of the diameter at the point of contact; and the abscisses DK, *dk* being equal, the tangents will cut off equal segments from PQ, and therefore will pass through the same point I.

Cor. 3. And if two diameters be at equal distances from a third, on opposite sides, and chords be drawn from the vertex of the middle diameter to the vertices of the other two, tangents drawn parallel to the chords will intersect each other in the middle diameter produced. For the semi-ordinates PK, P*k*, are the halves of chords so drawn, and DI, *d*I are tangents parallel to these chords.

PROPOSITION XVI.

Let CD, *any diameter of a parabola whose vertex is* C, *intersect a chord* AB *in* D; *from the ends of the chord inflect straight lines* AE, BE, *to* E, *any point in the curve, and let these cut the diameter in* H *and* K, *the point* H *being in* AE, *and* K *in* BE; *the segments* AD, BD *of the chord shall have the same ratio as the segments* CH, CK *of the diameter between its vertex and the inflected lines.*



From A, either extremity of the chord, draw AF parallel to the diameter CD, meeting BE in F.

By similar triangles, BF : BK = BA : BD,

and FE : KE = AF : HK;

therefore, taking the rectangles of corresponding terms of the ratios,

BF · FE : BK · KE = BA · AF : BD · HK.

But (Cor. 13) BF · FE : BK · KE = AF : KC = BA · AF : BA · KC (1, 6, E.);

therefore BD · HK = BA · KC;

and hence BA : BD = HK : KC;

and, by division, AD : BD = HC : KC.

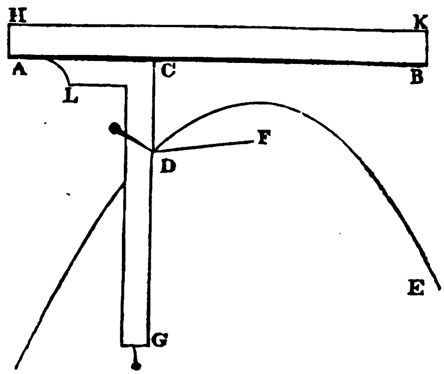
COR. Let BA be any chord in a parabola, and BI a tangent to the curve at one extremity of the chord; let any straight line DCL parallel to the axis meet the chord in D, the curve in C, and the tangent in L; the chord AB and the line DL will be similarly divided at D and C, that is, AD : DB = DC : CL.

Draw chords to E, any point in the curve, and let them meet DL in H and K: By the proposition AD : DB = HC : CK.

Suppose now that the point E moves along the curve until at last it come to B, the point of contact of the tangent; the line BK will then become BL, and AH will become AD, and the ratio of CH to CK will become the ratio of CD to CL; therefore AD : DB = CD : CL.

PROPOSITION XVII. PROBLEM.

The directrix and focus of a parabola being given by position, to describe the parabola by a mechanical construction.

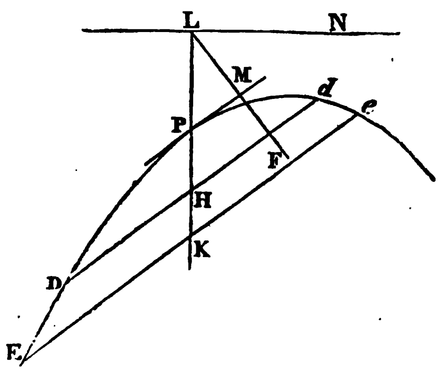


Let AB be the given directrix, and F the focus. Place the edge of the ruler ABKH along the directrix, and keep it fixed in that position. Let LCG be another ruler, of such a form that the side LC may slide along AB, the edge of the fixed ruler ABKH, and the part CG may have its edge CD constantly perpendicular to AB. Let GDF be a string of the same length as GC, the edge of the moveable ruler; let one end of the string be fixed at F, and the other fastened to G, a point in the moveable ruler. By means of the pin D let the string be stretched so that the part of it between G and D may be applied close to the edge of the moveable ruler, while at the same time the ruler slides along BA, the edge of the fixed ruler; the pin D will thus be constrained to move along CG, the edge of the ruler, and its point will trace upon the plane in which the directrix and focus are situated, a curve line

DE, which is the parabola required. For the string GDF being equal in length to GDC, if GD be taken from both, there remains DF equal to DC; that is, the distance of the moving point D from the focus is equal to its distance from the directrix, therefore the point D describes a parabola.

PROPOSITION XVIII. PROBLEM.

A parabola being given by position, to find its directrix and focus.

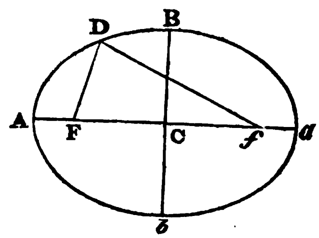


Let DPd be the given parabola; draw any two parallel chords Dd, Ee, and bisect them at H and K; join KH, meeting the parabola in P; the straight line PHK is a diameter (4 Cor. 8), the point P is its vertex, and Dd, Ee are ordinates to it. In HP produced take PL equal to one fourth part of a third proportional to PH and HD, and draw LN perpendicular to PL, the line LN will evidently be the directrix (12, and Def. 9). Draw PM parallel to the ordinates to the diameter PK, then PM will be a tangent to the curve at P (2 Cor. 8). Draw LM perpendicular to PM, and take MF = ML, and the point F will be the focus of the parabola (2 Cor. 3).

PART II.

OF THE ELLIPSE.

DEFINITIONS.



I. If two points F and f be given in a plane, and a point D be conceived to move around them in such a manner that $Df + DF$, the sum of its distances from them, is always the same; the point D will describe upon the plane a line ABab, which is called an *ellipse*.

II. The given points F, f are called the *foci* of the ellipse.

III. The point C which bisects the straight line between the foci is called the *centre*.

IV. The distance of either focus from the centre is called the *eccentricity*.

V. A straight line passing through the centre, and terminated both ways by the ellipse, is called a *diameter*.

VI. The extremities of a diameter are called its *vertices*.

VII. The diameter which passes through the foci is called the *transverse axis*, also the *greater axis*.

VIII. The diameter which is perpendicular to the transverse axis is called the *conjugate axis*, also the *lesser axis*.

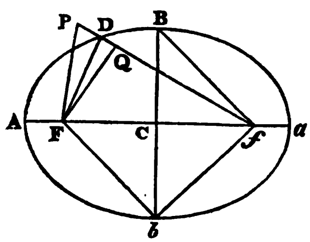
IX. Any straight line not passing through the centre, but terminated both ways by the ellipse, and bisected by a diameter, is called an *ordinate* to that diameter.

X. Each of the segments of a diameter intercepted between its vertices and an ordinate, is called an *absciss*.

XI. A straight line which meets the ellipse in one point only, and everywhere else falls without it, is said to *touch* the ellipse in that point, and is called a *tangent to the ellipse*.

PROPOSITION I.

If from any point in an ellipse two straight lines be drawn to the foci, their sum is equal to the transverse axis.



Let ABab be an ellipse, of which F, f are the foci, and Aa the transverse axis; let D be any point in the curve, and DF, Df lines drawn to the foci; $Df + DF = Aa$.

Because A, a are points in the ellipse, $Af + AF = aF + af$ (Def. 1),

therefore $Ff + 2\ AF = Ff + 2\ af$;

hence $2\ AF = 2\ af$, and $AF = af$,

and $Af + AF = Af + af = Aa$.

But D and A being points in the ellipse,

$Df + DF = Af + AF$, therefore $Df + DF = Aa$.

COR. 1. The sum of two straight lines drawn from a point without the ellipse to the foci is greater than the transverse axis. And the sum of two straight lines drawn from a point within the ellipse to the foci is less than the transverse axis.

Let PF, Pf be drawn from a point without the ellipse to the foci; let Pf meet the ellipse in D; join FD; then Pf + PF is greater than Df + DF (21, 1, E.), that is, than Aa. Again, let QF, Qf be drawn from a point within the ellipse; let Qf meet the curve in D, and join FD; Qf + QF is less than Df + DF (21, 1, E.), that is, than Aa.

COR. 2. A point is without or within the ellipse, according as the sum of two lines drawn from it to the foci is greater or less than the transverse axis.

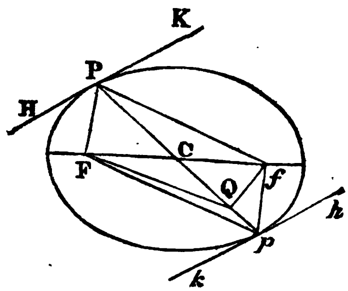
COR. 3. The transverse axis is bisected in the centre. Let C be the centre, then CF = Cf (Def. 3), and FA = fa, therefore CA = Ca.

COR. 4. The distance of either extremity of the conjugate axis from either of the foci is equal to half the transverse axis. Let Bb be the conjugate axis; join Fb, fb. Because CF = Cf, and Cb is common to the triangles CFb, Cfb, also the angles at C are right angles, these triangles are equal; hence Fb = fb, and since Fb + bf = Aa, Fb = AC.

COR. 5. The conjugate axis is bisected in the centre. Join fb, fB. By the last corollary Bf = bf; therefore the angles fBC, fbC are equal; now fC is common to the triangles fCB, fCb, and the angles at C are right angles, therefore (26, 1, E.) CB = Cb.

PROPOSITION II.

Every diameter of an ellipse is bisected in the centre.



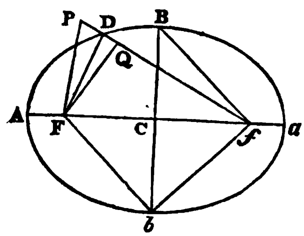
Let Pp be a diameter, it is bisected in C. For if Cp be not equal to CP, take CQ, equal to CP, and from the points P, p, Q, draw lines to F, f, the foci. The triangles FCP, fCQ having $FC = Cf$, $PC = CQ$, and the angles at C equal, are in all respects equal, therefore $FP = fQ$; in like manner it appears that $fP = FQ$, therefore $FQ + fQ$ is equal to $FP + fP$, or (Def. 1) to $Fp + fp$, which is absurd (21, 1, E.); therefore $PC = Cp$.

Cor. 1. Every diameter meets the ellipse in two points only.

Cor. 2. Every diameter divides the ellipse into two parts, which are equal and similar, the like parts of the curve being at opposite extremities of the diameter.

PROPOSITION III.

The square of half the conjugate axis of an ellipse is equal to the rectangle contained by the segments into which the transverse axis is divided by either focus.



Draw a straight line from f, either of the foci, to B, either of the extremities of the conjugate axis.

$$\text{Then } BC^2 + Cf^2 = Bf^2 = Ca^2 \text{ (4 Cor. 1).}$$

But because Aa is bisected at C,

$$Ca^2 = Af \cdot fa + Cf^2 \text{ (5, 2, E.)};$$

$$\text{therefore } BC^2 + Cf^2 = Af \cdot fa + Cf^2,$$

$$\text{and } BC^2 = Af \cdot fa.$$

PROPOSITION IV. PROBLEM.

To find any number of points in an ellipse, having given the transverse axis and foci.

Let F, f be the foci, Aa the transverse axis, and C the centre. Suppose the problem resolved, and that D is a point in the ellipse; join DF, Df; take AH in the axis equal to DF; then aH will be equal to Df (1).

And $HA - Ha = DF - Df$;

But $HA - Ha = HC + Ca - Ha = 2CH$;

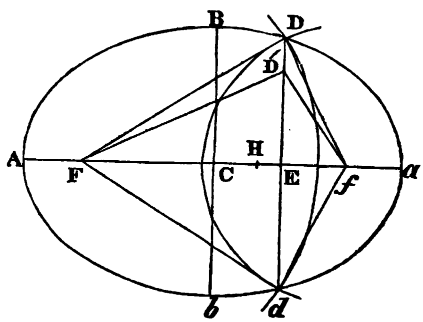
therefore $DF - Df = 2CH$.

Now $DF < Ff + Df$;

and hence $DF - Df \angle Ff$;

therefore $2CH < Ff$ and $CH \angle Cf$.

Thus it appears that the point H may be anywhere between the foci; but that it cannot be between the foci and the vertices.



Construction.—Take H, any point between the foci, and from F and f as centres, with the distances HA, Ha describe circles, which will cut each other in two points D, d, one on each side of the axis. These are points in the ellipse.

Join DF, Df, also dF, df. Because $DF + Df = HA + Ha = Aa$, therefore D is a point in the ellipse. In like manner it appears that d is in the ellipse.

In this way, by taking different points H, may any number of points in the ellipse be found.

Cor. I. Any perpendicular to the transverse axis between its extremities meets the ellipse in two points, and

in no more. For, if the perpendicular Dd could meet the curve in two points D, D′, on the same side of the axis, then DF, Df, also D′F, D′f, being drawn to the foci, DF + Df would be equal to D′F + D′f. Now, supposing D′ to be the point nearer to the axis, DF will be greater than D′F, and Df greater than D′f (19, 1, E.), and DF + Df greater than D′F + D′f; therefore D and D′ cannot both be points in the ellipse.

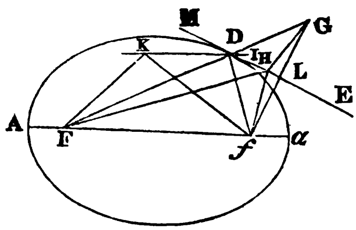
Cor. 2. Every chord Dd, in an ellipse, perpendicular to the transverse axis, is bisected by that axis, and therefore is an ordinate to it. For the chord Dd in the ellipse is also a chord in a circle, the centre of which is in the axis.

Cor. 3. Of all the straight lines that can be drawn from either focus to the curve, the longest is that which passes through the centre, and the shortest is the remainder of the transverse axis. And only two equal straight lines can be drawn from the focus to the curve, one on each side of the axis.

Scholium. From this proposition it appears that an ellipse is a curve, which returns into itself, thus enclosing a finite area; also, that the spaces between the curve and the axis on each side are alike in every way; so that if the ellipse were resolved into two portions, by cutting it along the axis, the space ABDa, if turned over, would coincide entirely with the space Abda. Now it has been shown that the same spaces will coincide if one of them be reversed (2 Cor. 2); and then the curve ABDa will coincide with adbA. Hence it follows that the two axes divide the whole ellipse into four portions exactly alike, and which, by superposition, may be applied on each other.

PROPOSITION V.

The straight line which bisects the angle adjacent to that which is contained by two straight lines drawn from any point in the ellipse to the foci, is a tangent to the curve in that point.



Let D be any point in the curve; let DF, D*f* be straight lines drawn to the foci; the straight line DE which bisects the angle *f*DG adjacent to *f*DF, is a tangent to the curve at D.

Take H any other point in DE; make DG = D*f*, and join H*f*, HF, HG, *f*G; let *f*G meet DE in L. Because D*f* = DG, and DL is common to the triangles D*f*L, DGL, and the angles *f*DL, GDL are equal, these triangles are equal, and *f*L = LG, and hence *f*H = HG (4, 1, E.), and FH + *f*H = FH + HG; but FH + HG is greater than FG, that is, greater than FD + *f*D or A*a*, therefore FH + *f*H is greater than A*a*; hence the point H is without the ellipse (2 Cor. 1), and therefore DHE is a tangent to the curve at D (Def. 11).

Cor. 1. There cannot be more than one tangent at the same point; for D is such a point in the line DE, that the sum of DF, D*f*, the distances of that point from the foci, is evidently less than the sum of HF, H*f*, the distances of H, any other point in that line; and if another line KDI be drawn through D, there is in like manner a point K in that line which will be different from D, such, that the sum of FK, *f*K is less than the sum of the dis-

tances of any other point in KI, and therefore less than FD + fD; therefore the point K will be within the ellipse (2 Cor. 1), and the line KI will cut the curve.

Cor. 2. A perpendicular to the transverse axis at either of its extremities is a tangent to the curve. The demonstration is the same as for the proposition, if it be considered that when D falls at either extremity of the axis, the point L falls also at the extremity of the axis; and thus the tangent DE, which is always perpendicular to fL, is perpendicular to the axis.

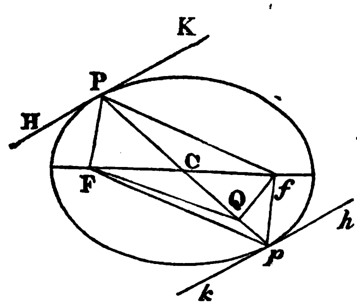
Cor. 3. A perpendicular to the conjugate axis at either of its extremities is a tangent to the curve. For the perpendicular evidently bisects the angle adjacent to that which is contained by lines drawn from the extremity to the foci.

Cor. 4. A tangent to the ellipse makes equal angles with straight lines drawn from the point of contact to the foci. For the angle fDE being equal to GDE, is also equal to FDM, which is vertical to GDE.

Scholium. From the property of the ellipse which forms this last corollary, the points F and f take the name of *Foci*. For writers on optics show that if a polished concave surface be formed, whose figure is that produced by the revolution of an ellipse about its transverse axis, rays of light which flow from one focus, and fall upon that surface, are reflected to the other focus; so that if a luminous point be placed in one focus, there is formed by reflection an image of it in the other focus.

PROPOSITION VI.

The tangents at the vertices of any diameter of an ellipse are parallel.



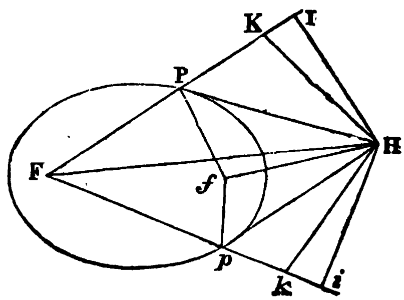
Let Pp be a diameter, and HPK, hpk tangents at its vertices; draw straight lines from P and p to F and f, the foci. The triangles FCP, fCp, having FC = fC, CP = Cp (2), and the angles at C equal, are in all respects equal; and because the angle FPC is equal to Cpf, FP is parallel to fp (27, 1, E.); therefore Pf is equal and parallel to pF (33, 1, E.); thus FPfp is a parallelogram, of which the opposite angles P and p are equal (34, 1, E.). Now the angles FPH, fph are evidently half the supplements of these angles (4 Cor. 5), therefore the angles FPH, fph are equal, and hence CPH, Cph are also equal, and consequently HP is parallel to hp.

Cor. 1. If tangents be drawn to an ellipse at the vertices of a diameter; straight lines drawn from either focus to the points of contact make equal angles with these tangents. For the angle Fpk is equal to FPH.

Cor. 2. The axes of an ellipse are the only diameters which are perpendicular to tangents at their vertices. For let Pp be any other diameter, then PF and pF are necessarily unequal, and therefore the angles FpP, FPp are also unequal; to these add the equal angles Fpk, FPH, and the angles Cpk, CPH are unequal; therefore neither of them can be a right angle (29, 1, E.).

PROPOSITION VII.

If a straight line be drawn from either focus of an ellipse to the intersection of two tangents to the curve; it will make equal angles with straight lines drawn from the same focus to the points of contact.



Let HP, Hp, tangents to an ellipse at P and p, intersect each other at H; draw PF, pF, HF to F, either of the foci; the line HF makes equal angles with FP, Fp.

Draw Pf, pf, Hf to f the other focus, and in FP, Fp produced take PK = Pf and pk = pf; join HK, Hk.

The triangles HPK, HPf have PK = Pf, PH common to both, and the angles KPH, fPH equal (5), therefore they are in every way equal (4, 1, E.), and have HK = Hf. In the same way it may be shown that the triangles Hpk, Hpf are in every way equal, and therefore that Hk = Hf.

The triangles HFK, HFk have HK = Hk (for each is equal to Hf), HF common to both, and FK = Fk, because each is equal to PF + Pf or pF + pf, that is, to the transverse axis; therefore they are in all respects equal, and the angle HFK is equal to the angle HFk; wherefore HF makes equal angles with FP and Fp.

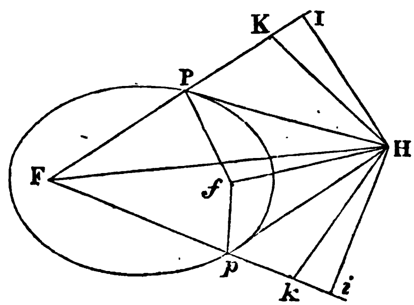
Cor. Perpendiculars drawn from the intersection of

two tangents to straight lines drawn from either focus through the points of contact, are equal.

Let HI, Hi be perpendiculars drawn from H, the intersection of the tangents PH, pH on the lines FP, Fp. The triangles HFI, HFi, are in all respects equal (26, 1, E.), therefore HI = Hi.

PROPOSITION VIII.

Straight lines drawn from the intersection of two tangents to the foci, make equal angles with the tangents.



Let F, f be the foci of an ellipse, and let straight lines HP, Hp, which intersect each other at H, touch the ellipse at P and p, also let HF, Hf be lines drawn to the foci; the angles PHF, pHf are equal.

The same construction being made as in Prop. VII. because the angles FHK, FHk are equal,

and $\mathrm{FHK} = \mathrm{FHP} + \mathrm{PHK} = \mathrm{FHP} + \mathrm{PH}f = 2\mathrm{FHP} + \mathrm{FH}f$,

and in like manner

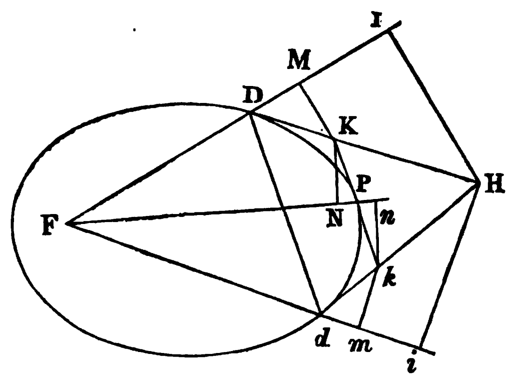
$\mathrm{FH}k = \mathrm{FH}p + p\mathrm{H}k = \mathrm{FH}p + p\mathrm{H}f = 2f\mathrm{H}p + \mathrm{FH}f$,

$$\text{therefore } 2\mathrm{FHP} + \mathrm{FH}f = 2f\mathrm{H}p + \mathrm{FH}f,$$

$$\text{and hence } 2\mathrm{FHP} = 2f\mathrm{H}p, \text{ and } \mathrm{FHP} = f\mathrm{H}p.$$

PROPOSITION IX.

If two tangents to an ellipse be at the extremities of a chord, and a third tangent be parallel to the chord; the part of this tangent intercepted by the other two is bisected at the point of contact.



Let HD, Hd be tangents at the extremities of the chord Dd, and KPk a tangent parallel to Dd, meeting the other tangents in K and k; the intercepted segment Kk is bisected at P, the point of contact.

From the points of contact D, P, d, draw lines to F, either of the foci, and from H, K, k, the intersections of the tangents, draw perpendiculars to the lines drawn from the points of contact to the focus, viz. HI, Hi perpendicular to DF, dF; and KM, KN perpendicular to FD, FP; and km, kn perpendicular to Fd, FP.

The triangles DHI, DKM are manifestly equiangular, also the triangles dHi, dkm;

therefore DH : DK = HI : KM (4, 6, E.),

$$dH : dk = Hi : km ;$$

but because Dd is parallel to Kk, a side of the triangle HKk,

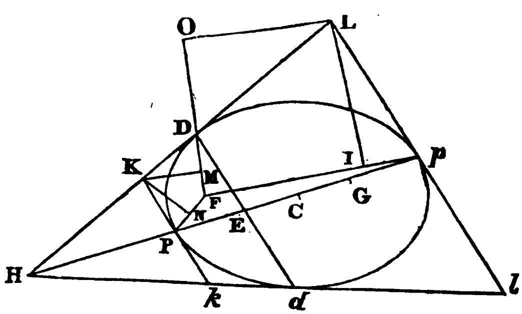
$$DH : DK = dH : dk \text{ (2, 6, E.)},$$

therefore $HI : KM = Hi : km$.

Now $HI = Hi$ (Cor. 7), therefore $KM = km$; but $KM = KN$ and $km = kn$ (Cor. 7), therefore $KN = kn$; and since from the similar triangles KPN, kPn, $KN : kn = KP : kP$, therefore KP is equal to kP.

PROPOSITION X.

Any chord parallel to a tangent, is bisected by the diameter that passes through the point of contact; or it is an ordinate to that diameter.



The chord DEd, which is parallel to Kk, a tangent at P, is bisected at E by the diameter PCp.

Draw Lpl, a tangent at p, the other end of the diameter, and DH, dH, tangents at D and d, the extremities of the chord, meeting the other tangents in K, k, and L, l: Then

KPk and Lpl are bisected at P and p (9); therefore the diameter Pp, when produced, will pass through H, and bisect Dd, which is parallel to Kk or Ll, in E. (Lemma to Prop. 8, Part I.)

Cor. 1. Straight lines which touch an ellipse in the extremities of an ordinate to any diameter, intersect each other in that diameter produced.

Cor. 2. Every ordinate to a diameter is parallel to a tangent at its vertex: for if not, let a tangent be drawn parallel to the ordinate; then the diameter drawn through the point of contact would bisect the ordinate; and thus the same line would be bisected in two different points, which is absurd.

Cor. 3. All the ordinates to the same diameter are parallel to each other.

Cor. 4. A straight line that bisects two parallel chords, and terminates in the curve, is a diameter.

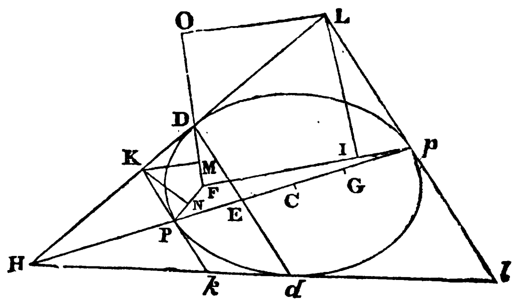
Cor. 5. The ordinates to either axis are perpendicular to that axis: and no other diameter is perpendicular to its ordinates.

PROPOSITION XI.

If a tangent to an ellipse meet a diameter, and from the point of contact an ordinate be drawn to that diameter; the semi-diameter will be a mean proportional between the segments of the diameter intercepted between the centre and the ordinate, and between the centre and the tangent.

Let DH, a tangent to the ellipse at D, meet the diameter Pp produced in H, and let DEd be an ordinate to that diameter; CE : CP = CP : CH.

Through P and p, the vertices of the diameter, draw the tangents PK, pL, meeting DH in K and L; draw PF, pF, to either of the foci, join DF, and draw KM and KN perpendicular to FD and FP, also LO and LI perpendicular to FD and Fp.



The triangles PKN, pLI, are equiangular; for the angles at N and I are right angles, and the angles NPK IpL are equal (1 Cor. 6); therefore

PK : pL = KN : LI (4, 6, E.) = KM : LO (Cor. 7).

But the triangles KDM, LDO, being manifestly equiangular,

KM : LO = KD : LD;

therefore PK : pL = KD : LD.

But because of the parallel lines PK, ED, pL, the triangles HPK, HpL, are equiangular; and the lines HL, Hp, are similarly divided in K, D, and in P, E (10, 6, E.), hence

PK : pL = HP : Hp, and KD : LD = PE : pE;

therefore HP : Hp = PE : pE.

Take CG = CE, and then PE = pG, and by conversion

HP : Pp = PE : EG;

and taking the halves of the consequents,

HP : PC = PE : EC;

and by composition, HC : PC = PC : EC.

Cor. 1. The rectangle $PE \cdot Ep$ is equal to the rectangle $HE \cdot EC$.

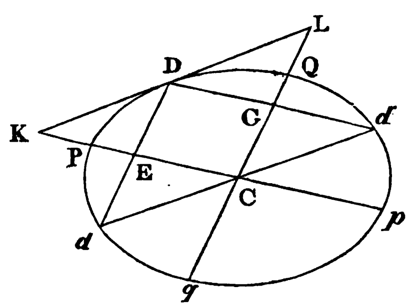
$$\text{For } PC^2 = HC \cdot CE \text{ (17, 6, E.)}$$
$$= HE \cdot EC + EC^2 \text{ (3, 2, E.)};$$
$$\text{also } PC^2 = PE \cdot Ep + EC^2 \text{ (5, 2, E.)};$$
$$\text{therefore } HE \cdot EC = PE, Ep.$$

Cor. 2. The rectangle $PH \cdot Hp$ is equal to the rectangle $EH \cdot HC$.

$$\text{For } HC^2 = PH \cdot Hp + CP^2 \text{ (6, 2, E.)};$$
$$\text{and } HC^2 = EH \cdot HC + EC \cdot HC \text{ (1, 2, E.)},$$
$$= EH \cdot HC + CP^2 \text{ (by the Prop.)};$$
$$\text{therefore } PH \cdot Hp = EH \cdot HC.$$

PROPOSITION XII.

If a diameter of an ellipse be parallel to the ordinates to another diameter; the latter diameter is parallel to the ordinates to the former.



Let Qq, a diameter of an ellipse, be parallel to DEd, any ordinate to the diameter Pp; the diameter Pp shall be parallel to the ordinates to the diameter Qq.

Draw the diameter dCd' through one extremity of the ordinate Dd, and join d' and D, the other extremity of the ordinate, meeting Qq in G. Because dd' is bisected in C (2), and CG is parallel to dD, the line Dd' is bisected at G (2, 6, E.); therefore Dd' is an ordinate to the diameter Qq (Def. 9), and because dd' and Dd are bisected at C and E, the diameter Pp is parallel to Dd' (2, 6, E.); therefore Pp is parallel to any ordinate to the diameter Qq.

DEFINITIONS.

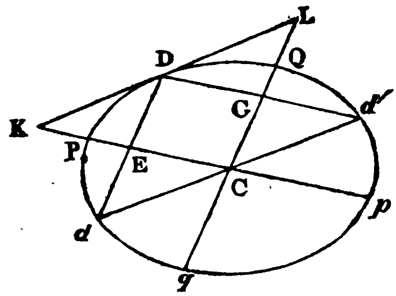
XII. Two diameters are said to be *conjugate* to one another when each is parallel to the ordinates to the other diameter.

Cor. Diameters which are conjugate to one another are parallel to tangents at the vertices of each other.

XIII. A third proportional to any diameter and its conjugate is called the *Parameter*, also the *Latus Rectum of that diameter.*

PROPOSITION XIII.

If an ordinate be drawn to any diameter of an ellipse; the rectangle contained by the segments of the diameter will be to the square of the semi-ordinate as the square of the diameter to the square of its conjugate.



Let DE*d* be an ordinate to the diameter P*p*, and let Q*q* be its conjugate,

$$PE \cdot Ep : DE^2 = Pp^2 : Qq^2.$$

Let KDL, a tangent at D, meet the diameter in K, and its conjugate in L: draw DG parallel to P*p*, meeting Q*q* in G. Because CP is a mean proportional between CE and CK (11),

$$CP^2 : CE^2 = CK : CE \text{ (2 Cor. 20, 6, E.)},$$

$$\text{and, by division, } CP^2 : PE \cdot Ep = CK : KE.$$

But because ED is parallel to CL,

$$CK : KE = CL : DE \text{ or } CG,$$

and because CQ is a mean proportional between CG and CL (11),

$$CL : CG = CQ^2 : CG^2 \text{ or } ED^2 \text{ (2 Cor. 20, 6, E.)},$$

$$\text{therefore } CP^2 : PE \cdot Ep = CQ^2 : DE^2,$$

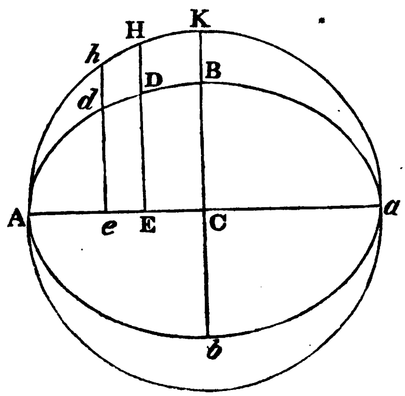
and, by inversion and alternation,

$$PE \cdot Ep : DE^2 = CP^2 : CQ^2 = Pp^2 : Qq^2.$$

Cor. 1. The squares of semi-ordinates and of ordinates to any diameter of an ellipse are to one another as the rectangles contained by the corresponding abscisses.

Cor. 2. The ordinates to any diameter, which intercept equal segments of that diameter from the centre, are equal to one another; and, conversely, equal ordinates intercept equal segments of the diameter from the centre.

Cor. 3. If a circle be described upon A*a*, either of the axes of an ellipse, as a diameter, and DE, *de*, any two semi-ordinates to the axis, meet the circle in H and *h*, DE shall be to *de* as HE to *he*.



$$\text{For } DE^2 : de^2 = AE \cdot Ea : Ae \cdot ea = HE^2 : he^2,$$

$$\text{therefore } DE : de = HE : h$$

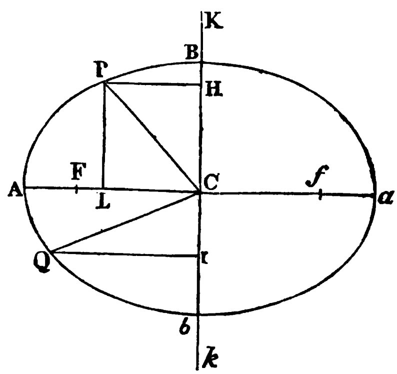
Cor. 4. If a circle be described on A*a* the transverse axis as a diameter, and DE, any ordinate to the axis, be produced to meet the circle in H; HE shall be to DE as the transverse axis A*a* to the conjugate axis B*b*. For, produce the conjugate axis to meet the circle in K, then, by last corollary,

$$HE : DE = KC, \text{ or } AC : BC = Aa : Bb.$$

Cor. 5. And if HE be divided at D, so that HE is to DE as the transverse axis to the conjugate axis, D is a point in the ellipse, and DE a semi-ordinate to the axis A*a*.

PROPOSITION XIV.

In Bb, *the conjugate axis of an ellipse, let there be taken on each side of the centre* C, *straight lines* CK, Ck, *each a fourth proportional to* CF, *the eccentricity, and* CA, CB, *half the transverse and conjugate axes: If then from* P, *a vertex of any diameter, there be drawn* PH *perpendicular to* Bb; *the square of the semi-diameter* PC *will have to the rectangle contained by the segments* KH, kH, *the constant ratio of the square of* CF *to the square of* CB.



Draw PL perpendicular to the transverse axis.

Because (13 of this, and 5,.2, E.)

$$CA^2 : CB^2 = CA^2 - CL^2 : PL^2,$$

and, by division,

$$CA^2 - CB^2 : CB^2 = CA^2 - (CL^2 + PL^2) : PL^2;$$

therefore (4 Cor. 1, and 47, 1, E.)

$$CF^2 : CB^2 = CA^2 - PC^2 : PL^2 \text{ or } CH^2.$$

But, by hypothesis, $CF^2 : CB^2 = CA^2 : CK^2$;

therefore $CA^2 : CK^2 = CA^2 - PC^2 : CH^2$,

and hence (19, 5, E.)

$$PC^2 : CK^2 - CH^2 = CA^2 : CK^2,$$

and (5, 2, E.)

$$PC^2 : KH \cdot Hk = CA^2 : CK^2 = CF^2 : CB^2.$$

Cor. 1. Hence the squares of any semi-diameters PC, QC, are to one another as the rectangles $KH \cdot Hk$, $KI \cdot Ik$, contained by the segments of the line Kk, between its extremities, and perpendiculars from the vertices of the diameters.

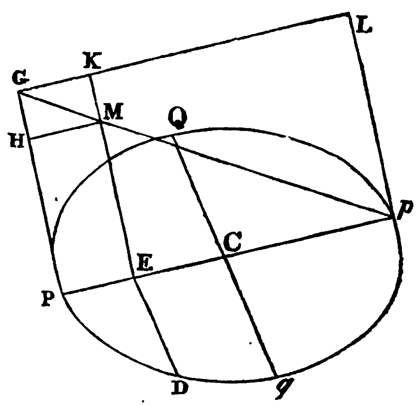
Cor. 2. The transverse axis is the greatest diameter, and the conjugate axis the least; and a diameter which is nearer to the transverse axis is greater than one more remote.

By hypothesis, $CF : CA = CB : CK$, therefore CK is greater than CB, and the points K, k, are without the ellipse: Suppose now a semi-diameter PC to turn about C, and that in every position PH is perpendicular to Kk: The rectangle $KH \cdot Hk$ will manifestly be greatest when PC coincides with AC, and least when it coincides with BC, and will decrease continually while PC passes from the position AC to BC; therefore the same will be true of the revolving semi-diameter PC, the square of which has a constant ratio to the rectangle $KH \cdot Hk$.

Cor. 3. Diameters which make equal angles with the transverse axis on opposite sides of it are equal; and only two equal diameters can be drawn, one on each side of the transverse axis.

PROPOSITION XV.

If an ordinate be drawn to any diameter of an ellipse, the rectangle under the abscisses of the diameter is to the square of the semi-ordinate as the diameter to its parameter.



Let DE be a semi-ordinate to the diameter Pp, let PG be the parameter of the diameter, and Qq the conjugate diameter. By the definition of the parameter (Def. 13),

$$\mathrm{P}p : \mathrm{Q}q = \mathrm{Q}q : \mathrm{PG},$$

therefore $\mathrm{P}p : \mathrm{PG} = \mathrm{P}p^2 : \mathrm{Q}q^2$ (2 Cor. 20, 6, E.).

But $\mathrm{P}p^2 : \mathrm{Q}q^2 = \mathrm{PE} \cdot \mathrm{E}p : \mathrm{DE}^2$ (13),

therefore $\mathrm{PE} \cdot \mathrm{E}p : \mathrm{DE}^2 = \mathrm{P}p : \mathrm{PG}$.

Cor. Let the parameter PG be perpendicular to the diameter Pp; join pG, and from E draw EM parallel to PG, meeting pG in M. The square of DE, the semi-ordinate, is equal to the rectangle contained by PE and EM.

For $\mathrm{PE} \cdot \mathrm{E}p : \mathrm{DE}^2 = \mathrm{P}p : \mathrm{PG}$,

and $\mathrm{P}p : \mathrm{PG} :: \mathrm{E}p : \mathrm{EM} = \mathrm{PE} \cdot \mathrm{E}p : \mathrm{PE} \cdot \mathrm{EM}$,

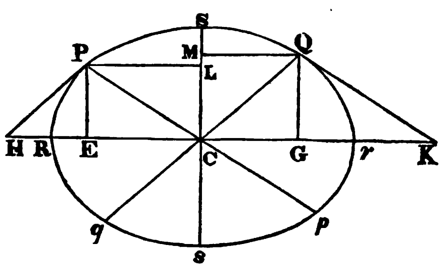
therefore $\mathrm{DE}^2 = \mathrm{PE} \cdot \mathrm{EM}$.

Scholium. If the rectangles PGLp, HGKM, be completed, it will appear that the square of ED is equal to the rectangle MP, which rectangle is less than the rectangle KP, contained by the absciss PE and parameter

PG, by a rectangle KH similar and similarly situated to LP, the rectangle contained by the diameter and parameter. It was on account of the deficiency of the square of the ordinate from the rectangle contained by the abscise and parameter that Apollonius called the curve line, to which the property belonged, an ellipse.

PROPOSITION XVI.

If from the vertices of two conjugate diameters of an ellipse there be drawn ordinates to any third diameter; the square of the segment of that diameter intercepted between either ordinate and the centre is equal to the rectangle contained by the segments between the other ordinate and the vertices of the same diameter.



Let Pp, Qq be two conjugate diameters, and PE, QG semi-ordinates to any third diameter Rr; $CG^2 = RE \cdot Er$, and $CE^2 = RG \cdot Gr$.

Draw the tangents PH, QK meeting Rr in H and K. The rectangles $HC \cdot CE$ and $KC \cdot CG$ are equal, for each is equal to CR^2 (11), therefore $HC : CK = CG : CE$.

But the triangles HPC, CQK are evidently similar (Cor. Def. 12), and PE being parallel to QG, their bases CH, KC are similarly divided at E and G, therefore

$HC : CK = HE : CG$, wherefore $CG : CE = HE : CG$,

consequently $CG^2 = CE \cdot EH =$ (1 Cor. 11) $RE \cdot Er$. In like manner it may be shown that $CE^2 = RG \cdot Gr$.

Cor. 1. Let Ss be the diameter that is conjugate to Rr, then Rr is to Ss as CG to PE, or as CE to QG.

$$\text{For } Rr^2 : Ss^2 = RE \cdot Er, \text{ or } CG^2 : PE^2;$$

$$\text{therefore } Rr : Ss :: CG : PE.$$

In like manner $Rr : Ss :: CE : QG$.

Cor. 2. The sum of the squares of CE, CG, the segments of the diameter to which the semi-ordinates PE, QG are drawn, is equal to the square of CR the semi-diameter.

$$\text{For } CE^2 + CG^2 = CE^2 + RE \cdot Er = CR^2 \text{ (5, 2, E.)}$$

Cor. 3. The sum of the squares of any two conjugate diameters is equal to the sum of the squares of the axes.

Let Rr, Ss be the axes, and Pp, Qq any two conjugate diameters; draw PE, QG perpendicular to Rr, and PL, QM perpendicular to Ss. Then

$$CE^2 + CG^2 = CR^2,$$

$$\text{and } CM^2 + CL^2, \text{ or } GQ^2 + PE^2 = CS^2,$$

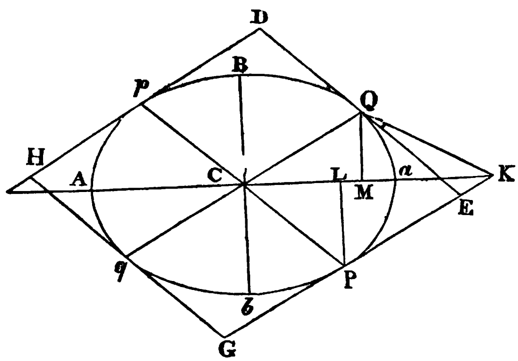
$$\text{therefore } CE^2 + PE^2 + CG^2 + GQ^2 = CR^2 + CS^2;$$

$$\text{that is (47, 1, E.), } CP^2 + CQ^2 = CR^2 + CS^2,$$

$$\text{therefore } Pp^2 + Qq^2 = Rr^2 + Ss^2.$$

PROPOSITION XVII.

If four straight lines be drawn touching an ellipse at the vertices of any two conjugate diameters; the parallelogram formed by these lines is equal to the rectangle contained by the transverse and conjugate axes.



Let Pp, Qq be any two conjugate diameters; a parallelogram DEGH formed by tangents to the curve at their vertices is equal to the rectangle contained by Aa, Bb, the two axes.

Produce Aa, one of the axes, to meet the tangent PE in K; join QK, and draw PL, QM perpendicular to Aa.

Because CK : CA = CA : CL (11),

and CA : CB = CL : QM (1 Cor. 16),

ex æq. CK : CB = CA : QM,

therefore CK · QM = CB · CA (16, 6, E.).

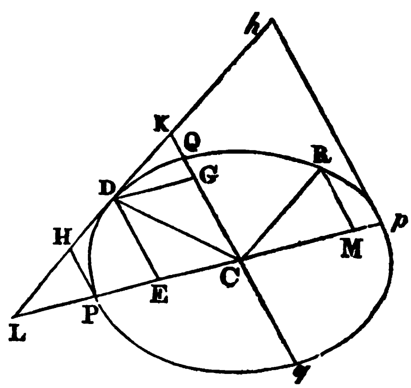
But CK · QM = twice trian. CKQ = paral. CPEQ (41, 1, E.),

therefore the parallelogram CPEQ = CB · CA,

and taking the quadruples of these, the parallelogram DEGH is equal to the rectangle contained by Aa and Bb.

PROPOSITION XVIII.

If two tangents at the vertices of any diameter of an ellipse meet a third tangent; the rectangle contained by their segments between the points of contact and the points of intersection is equal to the square of the semi-diameter to which they are parallel: and the rectangle contained by the segments of the third tangent between its point of contact and the parallel tangents is equal to the square of the semi-diameter to which it is parallel.



Let PH, ph, tangents at the vertices of a diameter Pp, meet $\text{HD}h$, a tangent to the curve at any point D, in H and h: let CQ be the semi-diameter to which the tangents PH, ph are parallel, and CR that to which $\text{H}h$ is parallel:

$$\text{PH} \cdot ph = \text{CQ}^2, \text{ and } \text{DH} \cdot \text{D}h = \text{CR}^2.$$

If the tangent $\text{HD}h$ be parallel to Pp, the proposition is manifest. If it be not parallel, let it meet the semi-diame-

ters CP, CQ, in L and K. Draw DE, RM parallel to CQ, and DG parallel to CP.

$$\text{Because } LP \cdot Lp = LE \cdot LC \text{ (2 Cor. 11)},$$

$$LP : LE = LC : Lp;$$

hence, and because of the parallels PH, ED, CK, *ph*,

$$PH : ED = CK : ph;$$

$$\text{wherefore } PH \cdot ph = ED \cdot CK.$$

$$\text{But } ED \cdot CK = CG \cdot CK = CQ^2 \text{ (11)},$$

$$\text{therefore } PH \cdot ph = CQ^2.$$

Again, the triangles LED, CMR are evidently similar, and LD, LE similarly divided at H and P, also at *h* and *p*,

$$\text{therefore } PE : HD = (LE : LD =)\ CM : CR,$$

$$\text{also } pE : hD = (LE : LD =)\ CM : CR;$$

hence, taking the rectangles of the corresponding terms,

$$PE \cdot pE : HD \cdot hD :: CM^2 : CR^2 \text{ (3 Cor. 20, 6, E.)}.$$

But if CD be joined, the points D and R are evidently the vertices of two conjugate diameters (Cor. Def. 12), and therefore $PE \cdot pE = CM^2$ (16);

$$\text{therefore } HD \cdot hD = CR^2.$$

COR. The rectangle contained by LD and DK, the segments of a tangent intercepted between D the point of contact, and P*p*, Q*q*, any two conjugate diameters, is equal to the square of CR, the semi-diameter to which the tangent is parallel.

Let the parallel tangents PH, *ph* meet LK in H and *h*, and draw DE a semi-ordinate to P*p*. Because of the parallels PH, ED, CK, *ph*,

$$LE : LD = EP : DH,$$

$$\text{and } EC : DK :: Ep : Dh;$$

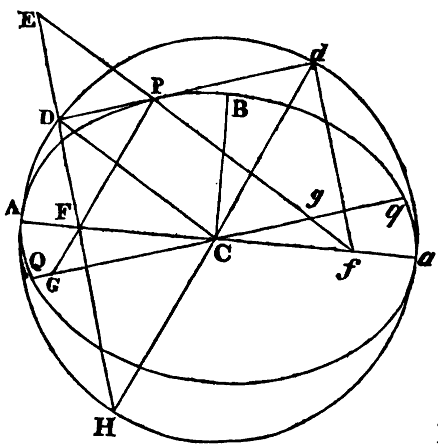
$$\text{therefore } LE \cdot EC : LD \cdot DK :: EP \cdot Ep : DH \cdot Dh.$$

$$\text{But } LE \cdot EC = EP \cdot Ep \text{ (1 Cor. 11)},$$

$$\text{therefore } LD \cdot DK = DH \cdot Dh = \text{(by this Prop.) } CR^2.$$

PROPOSITION XIX.

If two straight lines be drawn from the foci of an ellipse perpendicular to a tangent; straight lines drawn from the centre to the points in which they meet the tangent will each be equal to half the transverse axis.



Let DPd be a tangent to the curve at P, and FD, fd perpendiculars to the tangent from the foci; the straight lines joining the points C, D, and C, d, are each equal to AC, half the transverse axis.

Join FP, fP, and produce FD, fP till they intersect in E. The triangles FDP, EDP have the angles at D right angles, and the angles FPD, EPD equal (4 Cor. 5), and the side DP common to both; they are therefore equal, and consequently have ED = DF, and EP = PF, therefore Ef = FP + Pf = Aa. Now the straight lines FE, Ff, being bisected at D and C, the line DC is parallel to Ef, and thus the triangles FCD, FfE are similar:

therefore FC : CD = Ff : fE or Aa.

But FC is half of Ff, therefore CD is half of Aa. In like manner it may be shown that Cd is half of Aa.

Cor. If the diameter Qq be drawn parallel to the tangent Dd, it will cut off from PF, Pf the segments PG, Pg,

each equal to AC half the transverse axis. For CdPG, CDPg are parallelograms, therefore PG $=d$C $=$ AC, and Pg $=$DC $=$ AC.

PROPOSITION XX.

The rectangle contained by perpendiculars drawn from the foci of an ellipse to a tangent is equal to the square of half the conjugate axis.

Let DPd (fig. Prop. 19) be a tangent, and FD, fd, perpendiculars from the foci; the rectangle contained by FD and fd is equal to the square of CB half the conjugate axis.

It is evident, from the last proposition, that the points D, d are in the circumference of a circle whose centre is the centre of the ellipse, and radius CA half the transverse axis; now FDd being a right angle, if dC be joined, the lines DF, dC, when produced, will meet at H, a point in the circumference; and since FC $= f$C, and CH $=$ Cd, and the angles FCH, fCd are equal, FH is equal to fd; therefore

$$DF \cdot df = DF \cdot FH = AF \cdot Fa \text{ (35, 3, E.)} = CB^2 \text{ (3).}$$

Cor. If PF, Pf be drawn from the point of contact to the foci, the square of FD is a fourth proportional to fP, FP and BC². For the lines fP, FP make equal angles with the tangent (4 Cor. 5), and fdP, FDP are right angles, therefore the triangles fPd, FPD are similar, and

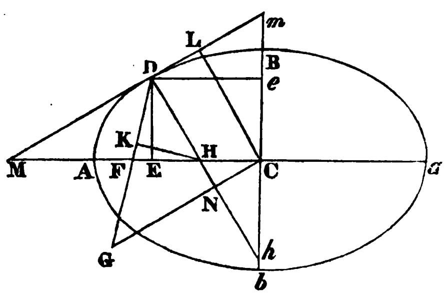
$$fP : FP = fd : FD = fd \cdot FD \text{ or } CB^2 : FD^2.$$

DEFINITION.

XIV. A perpendicular to a tangent to the curve at the point of contact is called a *Normal* to the curve.

PROPOSITION XXI.

If from C *the centre of an ellipse a straight line* CL *be drawn perpendicular to a tangent* LD, *and from* D *the point of contact a normal be drawn meeting the transverse axis in* H *and the conjugate axis in* h *; the rectangle contained by* CL *and* DH *is equal to the square of* CB, *the semi-conjugate axis ; and the rectangle contained by* CL *and* D*h* *is equal to the square of* CA, *the semi-transverse axis.*



Produce the axes to meet the tangent in M and m, and from D draw the semi-ordinates DE, De, which will be perpendicular to the axes.

The triangles DEH, CLm are equiangular, therefore DH : DE = Cm : CL; hence CL · DH = DE · Cm.

But DE · Cm, or Ce · Cm = BC² (11),

therefore CL · DH = BC².

In the same way it is shown that CL · Dh = AC².

Cor. 1. The segments DH, Dh of a normal intercepted between the point of contact and the axes, are to each other reciprocally as the squares of the axes by which they are terminated.

For AC² : BC² : : CL · Dh : CL · DH : : Dh : DH.

Cor. 2. If DF be drawn to either focus, and HK be drawn perpendicular to DF, the straight line DK shall be equal to half the parameter of the transverse axis.

Draw CG parallel to the tangent at D, meeting DH in

N, and DF in G. The triangles GDN, HDK are similar, therefore GD : DN = HD : DK,

and hence GD · DK = HD · DN.

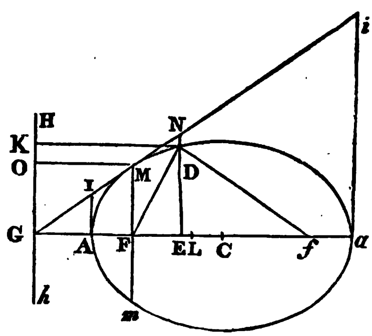
But GD = AC (Cor. 19), and ND = CL,

therefore AC · DK = HD · CL = (by the Prop.) CB^2;

wherefore AC : BC = BC : DK;

hence DK is half the parameter of A*a* (Def. 13).

DEFINITION.



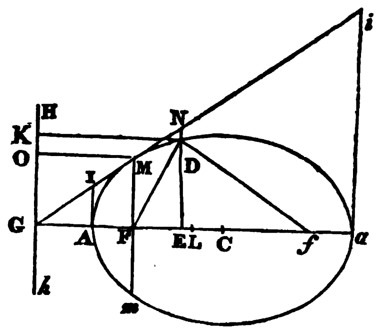
XV. If a point G be taken in the transverse axis of an ellipse produced, so that the distance of G from the centre may be a third proportional to CF the eccentricity, and CA the semi-transverse axis; a straight line HG*h*, drawn through G perpendicular to the axis, is called the *directrix of the ellipse.*

Cor. 1. If MF*m*, an ordinate to the axis, be drawn through the focus; tangents to the ellipse at the extremities of the ordinate will meet the axis at the point G (11).

Cor. 2. The ellipse has two directrices, for the point G may be taken on either side of the centre.

PROPOSITION XXII.

The distance of any point in an ellipse from either directrix is to its distance from the focus nearest that directrix in the constant ratio of the semi-transverse axis to the eccentricity.



Let D be any point in the ellipse, let DK be drawn perpendicular to the directrix, and let DF be drawn to the focus nearest the directrix; DK is to DF as CA, half the transverse axis, to CF, the eccentricity.

Draw D*f* to the other focus, and DE perpendicular to A*a*; take L a point in the axis, so that AL = FD, and consequently L*a* = D*f*; then CL is evidently half the difference between AL and *a*L, or FD and *f*D, and CE half the difference between *f*E and FE; and because D*f*+DF : *f*F = *f*E — FE : D*f*— DF (K, and 16 of 6, E.).

By taking the halves of the terms of the proportion,

CA : CF = CE : CL.

But CA : CF = CG : CA (Def. 15),

therefore CG : CA = CE : CL ;

hence (19, 5, E.) EG : AL = CG : CA = CA : CF,

that is, DK : DF = CA : CF.

Cor. 1. If the tangent GMN be drawn through M, the extremity of the ordinate passing through the focus, and ED be produced to meet GM in N, EN shall be equal to DF. For draw MO perpendicular to the directrix, then, because M and D are points in the ellipse,

$$FM : FD = MO : DK = FG : EG.$$

But the triangles GFM, GEN being similar,

$$FG : EG = FM : EN;$$

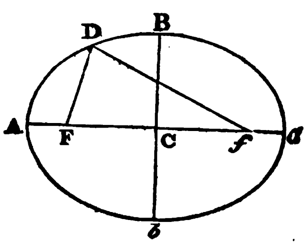
therefore $FM : FD = FM : EN$, and hence $FD = EN$.

COR. 2. If AI and ai be drawn perpendicular to the transverse axis at its extremities, meeting the tangent GM in I and i, then $AI = AF$ and $ai = a$F.

For $GA : AF = OM : MF = GF : MF = GA : AI$, therefore $AF = AI$; and, in like manner, it may be shown that $a\text{F} = ai$.

PROPOSITION XXIII. PROBLEM.

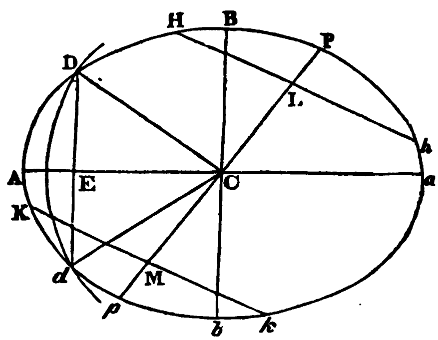
Two unequal straight lines which bisect each other at right angles being given by position; to describe an ellipse of which these may be the two axes, by a mechanical construction.



Let Aa be the transverse, and Bb the conjugate axes. About either extremity of the conjugate axis as a centre, with a radius equal to CA, half the transverse axis, describe arcs cutting that axis in F and f; these points will be the foci (4 Cor. 1). Let the ends of a string equal in length to Aa be fastened at the points F, f, and let the string be stretched by a pin at D, and while it is kept uniformly tense, let the point of the pin be carried along the plane about the centre C, till it return to the place from whence it set out. By this motion the point of the pin will trace on the plane a curve which will be the ellipse required, as is evident from the definition of the ellipse.

PROPOSITION XXIV.

An ellipse being given by position; to find its axes.



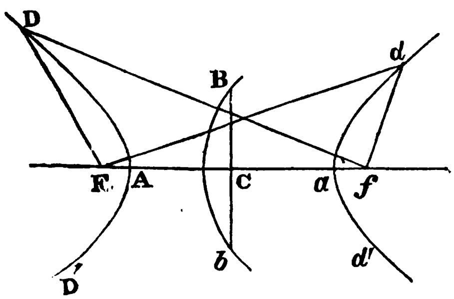
Let ABab be the given ellipse; draw two parallel chords Hh, Kk, and bisect them at L and M; join LM, and produce it to meet the ellipse in P and p, then Pp is a diameter (4 Cor. 10). Bisect Pp in C; the point C is the centre of the ellipse (2).

Take D any point in the ellipse, and on C as a centre, with the distance CD, describe a circle. If this circle be wholly without the curve, then CD must be half the transverse axis; but if it be wholly within the curve, then CD must be half the conjugate axis (14). If the circle neither be wholly without nor wholly within the ellipse, let the circle meet it again in d. Join Dd and bisect Dd in E; join CE, and produce CE to meet the ellipse in A and a: then Aa will be one of the axes (5 Cor. 10); for it is perpendicular to the line Dd (3, 3, E.), which is an ordinate to Aa; the other axis Bb will be found by drawing a straight line through the centre perpendicular to aA.

PART III.

OF THE HYPERBOLA.

DEFINITIONS.



I. If two points F, f be given in a plane, and a point D be conceived to move in such a manner that Df—DF, the difference of its distances from them, is always the same; the point D will describe upon the plane a line DAD′ called an *Hyperbola*. By assuming first one of the given points F, and then the other f, as that to which the moving point is nearest, the difference of the lines DF and Df in both cases being the same, there will be two hyperbolas DAD′, *dad′* described, opposite to each other, which are therefore called *Opposite Hyperbolas*.

COR. The lines DF, Df may become greater than any given line, therefore the hyperbolas extend to a greater distance from the given points F, f than any which can be assigned.

II. The given points F, f are called the *Foci of the hyperbola.*

III. The point C, which bisects the straight line between the foci, is called the *Centre.*

IV. The distance of either focus from the centre is called the *Eccentricity.*

V. A straight line passing through the centre, and terminated by the opposite hyperbolas, is called a *Transverse Diameter.* It is also sometimes called simply a *Diameter.*

VI. The extremities of a diameter are called its *Vertices.*

VII. The diameter which passes through the foci is called the *Transverse Axis.*

Cor. The vertices of the transverse axis lie between the foci. Let A be either of the vertices, then, because any side of a triangle is greater than the difference between the other two sides, Ff is greater than fD—DF, which is equal to fA—FA (Def. 1). Now this can only be true when A is between F and f.

VIII. A straight line Bb passing through the centre, perpendicular to the transverse axis, and limited at B and b by a circle described on one extremity of that axis, with a radius equal to the distance of either focus from the centre, is called the *Conjugate Axis.* It is also called the *Second Axis.*

Cor. The conjugate axis is bisected in the centre. This appears from 3, 3, E.

IX. Any straight line terminated both ways by the hyperbola, and bisected by a transverse diameter produced, is called an *Ordinate* to that diameter.

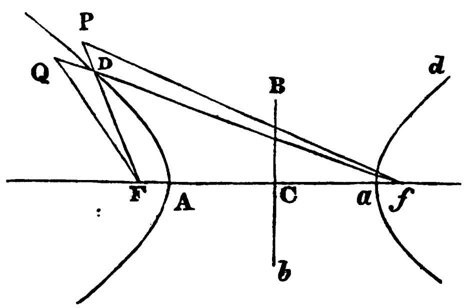
X. Each of the segments of a transverse diameter pro-

duced, intercepted by its vertices and an ordinate, is called an *Absciss.*

XI. A straight line which meets the hyperbola in one point only, and which everywhere else falls without the opposite hyperbolas, is said to *touch* the hyperbola in that point, and is called a *Tangent to the hyperbola.*

PROPOSITION I.

If from any point in an hyperbola two straight lines be drawn to the foci, their difference is equal to the transverse axis.



Let DA, *da* be opposite hyperbolas, of which F, f are the foci and Aa the transverse axis; let D be any point in the curve, and DF, Df lines drawn to the foci;

$$Df - DF = Aa.$$

Because A and a are points in the hyperbola,

$$Af - AF = aF - af \text{ (Def. 1)},$$
$$\text{therefore } Ff - 2AF = Ff - 2af;$$
$$\text{hence } 2AF = 2af, \text{ and } AF = af,$$
$$\text{and } Af - AF = Af - af = Aa.$$

But D and A being points in the hyperbola,
Df—DF = Af—AF, therefore Df—DF = Aa.

Cor. 1. The difference of two straight lines drawn from a point without the opposite hyperbolas to the foci is less than the transverse axis, and the difference of two straight lines drawn from a point within either of them to the foci is greater than the transverse axis.

Let Pf, PF be lines drawn from a point without the hyperbolas, that is, between the curve and its conjugate axis. The line PF must necessarily meet the curve; let D be the point of intersection; Pf is less than $PD + Df$ (20, 1, E.), therefore $Pf - PF$ is less than $(PD + Df) - PF$, that is, less than $Df - DF$, or Aa. Again, let Qf, QF be lines drawn from a point within either of the hyperbolas, Qf must necessarily meet the curve; let D be the point of intersection, join FD; QF is less than $QD + DF$, and therefore $Qf - QF$ is greater than $Qf - (QD + DF)$, that is, greater than $Df - DF$ or Aa.

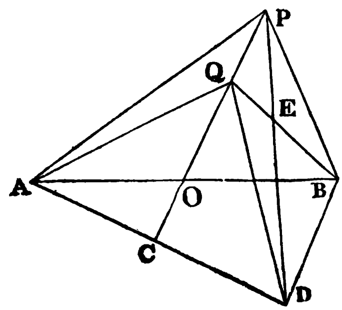
Cor. 2. A point is without or within the hyperbolas, according as the difference of two lines drawn from that point to the foci is less or greater than the transverse axis.

Cor. 3. The transverse axis is bisected in the centre. Let C be the centre; then $CF = Cf$ (Def. 3), and $FA = fa$, therefore $CA = Ca$.

Lemma I.

Let APB be a triangle, of which the side PA is greater than the side PB; draw a straight line from P, the vertex, to O, the middle of the base AB, and straight lines AQ, BQ, to any point in PO; the line QA will be greater than the line QB; and the excess of PA above PB will be greater than the excess of QA above QB.

Draw AC perpendicular to PO, and BD parallel to it, meeting AC in D; and join QD, PD, this last line meeting QB in E.



The triangles AOP, BOP have AO = BO, PO common to both, and PA greater than PB, therefore the angle AOQ is greater than the angle BOQ (25, 1, E.); and hence again, in the triangles AOQ, BOQ, the line AQ will be greater than BQ (24, 1, E.).

And because CO is parallel to DB, and AO = OB, therefore AC = CD (2, 6, E.). The triangles ACP, DCP have thus AC = DC, CP common, and the angle ACP equal to DCP, therefore PA = PD; and in the same way it appears that QA = QD.

And since PA is greater than PB, and QA than QB, therefore PD is greater than PB, and QD than QB.

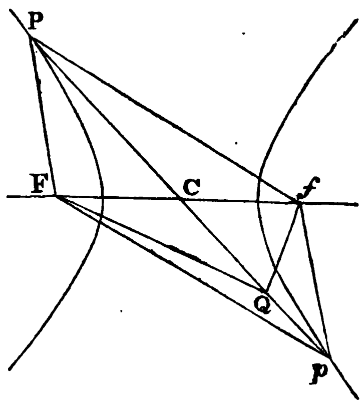
$$\text{Again, since } DE + EQ > DQ,$$
$$\text{therefore } DE + EQ - QB > DQ - QB;$$
$$\text{that is, } DE - EB > AQ - QB;$$
$$\text{also, since } PB < PE + EB,$$
$$\text{therefore } DP - PB > DP - PE - EB;$$
$$\text{that is, } AP - PB > DE - EB.$$

Now it was shown that $DE - EB > AQ - QB$;

much more then is $AP - PB > AQ - QB$.

PROPOSITION II.

Every transverse diameter of an hyperbola is bisected in the centre.



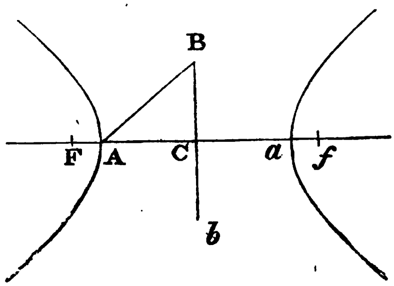
Let Pp be a transverse diameter; it is bisected in C; for if CP and Cp be unequal, take CQ equal to CP; from the points P, p, Q, draw straight lines to F and f the foci. The triangles PCF, QCf have CP equal to QC and CF equal to Cf (Def. 3), and the angles at C equal, therefore they are in every way equal, and PF is equal to Qf. In the same way, it appears that the triangles PCf, QCF are equal, and that Pf is equal to QF; therefore Pf — PF = QF — Qf; but because P and p are points in the hyperbola Pf — PF = pF — pf, therefore pF — pf = QF — Qf. Now this is impossible, because, by the preceding Lemma, pF — pf 7 QF — Qf; therefore CP and Cp are not unequal, that is, they are equal.

Cor. 1. Every transverse diameter meets the hyperbola in two points only.

Cor. 2. Every transverse diameter divides the opposite hyperbolas into two parts which are equal and similar, the like parts being at opposite parts of the diameter. From which it follows that the opposite hyperbolas may be applied one upon the other, so as entirely to coincide.

PROPOSITION III.

The square of half the conjugate axis of an hyperbola is equal to the rectangle contained by the straight lines between either focus and the extremities of the transverse axis.

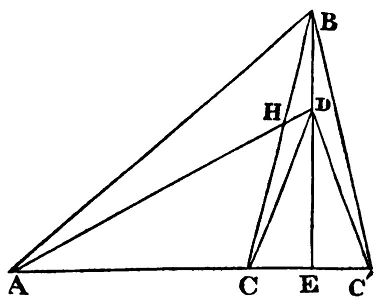


Draw a straight line from A, either of the extremities of the transverse axis, to B, either extremity of the conjugate axis. Then, $BC^2+CA^2=BA^2$ (47, 1, E.) $=CF^2$ (Def. 8). But because Aa is bisected at C and produced to F,

$$CF^2=AF\cdot Fa+CA^2 \text{ (6, 2, E.)},$$
$$\text{therefore } BC^2+CA^2=AF\cdot Fa+CA^2;$$
$$\text{and } BC^2=AF\cdot Fa.$$

Lemma II.

Let ABC be a triangle, of which the side BA is greater than the side BC; draw BE perpendicular to the side AC, and straight lines AD, CD to any point in BE; the line DA will be greater than the line DC; and the excess of DA above DC will be greater than the excess of BA above BC.



First, let the perpendicular BE fall without the triangle ABC, and let AD meet BC in H.

The angle DCA is greater than BCA; but BCA being an obtuse angle, is greater than BAC, which is acute; and again BAC is greater than DAC; much more then is DCA greater than DAC; therefore DA is greater than DC (19, 1, E.).

$$\text{And since } AH + HB > AB,$$
$$\text{therefore } AH + HB - BC > AB - BC;$$
$$\text{that is, } AH - HC > AB - BC.$$
$$\text{Again, since } CD < DH + CH,$$
$$\text{therefore } AD - CD > AD - DH - CH;$$
$$\text{that is, } AD - CD > AH - CH.$$

But it was shown that $AH - CH > AB - BC$;

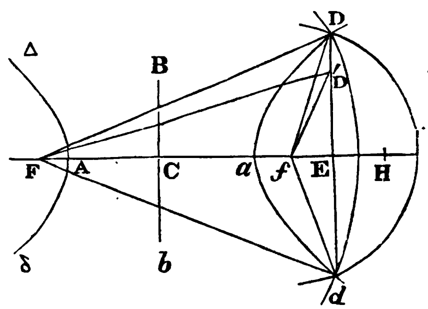
much more then is $AD - DC > AB - BC$.

Next, let the perpendicular fall within the triangle ABC'; take $EC=EC'$, and join BC, DC; then $BC=BC'$, and $DC=DC'$, therefore $DA > DC'$,

$$\text{and } BA-BC=BA-BC',$$
$$\text{also } DA-DC=DA-DC';$$
$$\text{but } DA-DC > BA-BC,$$
$$\text{therefore } DA-DC' > BA-BC'.$$

PROPOSITION IV. PROBLEM.

To find any number of points in an hyperbola, having given the transverse axis and foci.



Let F, f be the foci, Aa the transverse axis, and C the centre. Suppose the problem resolved, and that D is a point in the hyperbola. Join DF, Df. Take AH in the axis equal to DF; then aH will be equal to Df (Def. 1),

$$\text{and } HA+Ha=DF+Df;$$
$$\text{but } HA+Ha=HC+Ca+Ha=2CH,$$
$$\text{therefore } DF+Df=2CH.$$
$$\text{Now } DF+Df > Ff,$$
$$\text{therefore } 2CH > Ff \text{ and } CH > Cf.$$

Thus it appears that the point H cannot be between the foci F, f, and that it may be anywhere in the line Ff, produced both ways.

CONSTRUCTION.—Take H, any point in the axis produced both ways, except between F and f the foci, and from F and f as centres, with the distances HA, Ha, describe circles which will cut each other in two points D, d, one on each side of the axis. These are points in the hyperbola.

Join DF, Df, also dF, df. Because DF — Df = HA —Ha=Aa, therefore D is a point in the hyperbola; and in like manner it appears that d is a point in the hyperbola.

In this way may any number of points in the hyperbola be found.

COR. 1. Any perpendicular to the transverse axis which meets it produced either way, will cut the curve in two points, and in no more. For if the perpendicular Dd could meet the curve in two points D, D′, on the same side of the axis, then DF, Df, also D′F, D′f, being drawn to the foci, DF—Df would be equal to D′F — Df, which is impossible (Lemma 2).

COR. 2. Every chord Dd in an hyperbola, perpendicular to the transverse axis, is bisected by that axis, and therefore is an ordinate to it.

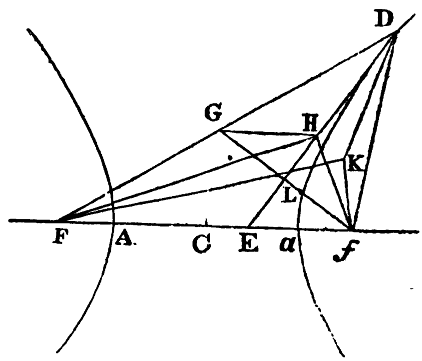
COR. 3. Of all the straight lines which can be drawn from either focus to either of the opposite hyperbolas, the shortest is that which passes through the centre (being produced if necessary); and only two equal straight lines can be drawn from either focus to one of the opposite hyperbolas, viz. one line on each side of the centre.

SCHOLIUM. From this proposition it appears that the

opposite hyperbolas recede continually from the foci and from the axis, and that they are entirely separated from each other, their nearest approach being at the vertices of the transverse axis. Also, that if the space D*ad*, bounded by either, were resolved into two by cutting it along the axis CH, the portions on each side of the axis would entirely coincide if one were turned over on the other. Now it was shown that the opposite hyperbolas might be applied one upon the other, viz. the curve D*ad* on δAΔ (2 Cor. 2), therefore the transverse axis divides the opposite hyperbolas into four spaces, indefinite in extent, but which are exactly alike, and may be placed one on another, so as entirely to coincide.

PROPOSITION V.

The straight line which bisects the angle contained by two straight lines drawn from any point in the hyperbola to the foci, is a tangent to the curve at that point.



Let D be the point in the curve; let DF, Df be straight lines drawn to the foci; the straight line DE which bisects the angle fDF, is a tangent to the curve.

Take H any other point in DE; take DG = Df, and join Hf, HF, HG, fG; let fG meet DE in L. Because Df = DG, and DL is common to the triangles DfL, DGL, and the angles fDL, GDL are equal, these triangles are equal, and fL = LG, and hence fH = HG (4, 1, E.); and FH — fH = FH — HG: but since FH is less than FG + GH, FH — HG is less than FG, that is, less than FD — fD or Aa, therefore FH — fH is less than Aa; hence the point H is without the hyperbola (2 Cor. 1), and consequently DHL is a tangent to the curve at D (Def. 11).

Cor. 1. There cannot be more than one tangent to the

hyperbola at the same point. For D is such a point in the line DE, that the difference of the lines DF, Df, the distances of that point from the foci, is greater than the difference of FH, fH, the distances of H, any other point in that line; and if another line KD be drawn through D, there is in like manner a point K in that line which will be different from D, such that the difference of FK, fK is greater than the difference of the distances of any other point in KD, and therefore greater than FD —fD; therefore the point K will be within the hyperbola (2 Cor. 1), and the line KD will cut the curve.

Cor. 2. A perpendicular to the transverse axis at either of its extremities is a tangent to the curve. The demonstration is the same as for the proposition, if it be considered that when D falls at either extremity of the axis, the point L falls also at the extremity of the axis, and thus the tangent DE, which is always perpendicular to fL, is perpendicular to the axis.

Cor. 3. Every tangent to either of the opposite hyperbolas passes between that hyperbola and the centre. Let the tangent DL meet the axis in E. Because DE bisects the angle FDf,

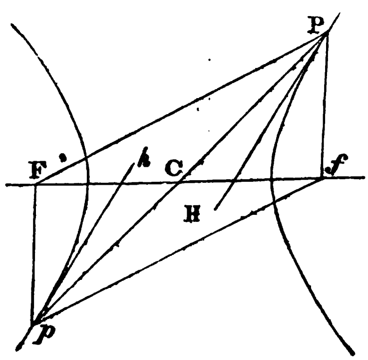
$$FD : fD :: FE : fE \ (3, 6, E.).$$

But FD is greater than fD (Def. 1), therefore FE is greater than fE, and hence E is between C and the vertex of the hyperbola to which DE is a tangent.

Scholium. From the property of the hyperbola which forms this proposition, the points F and f are called *foci;* for rays of light proceeding from one focus, and falling upon a polished surface whose figure is that formed by the revolution of the curve about the transverse axis, are reflected in lines passing through the other focus.

PROPOSITION VI.

The tangents at the vertices of any transverse diameter of an hyperbola are parallel.



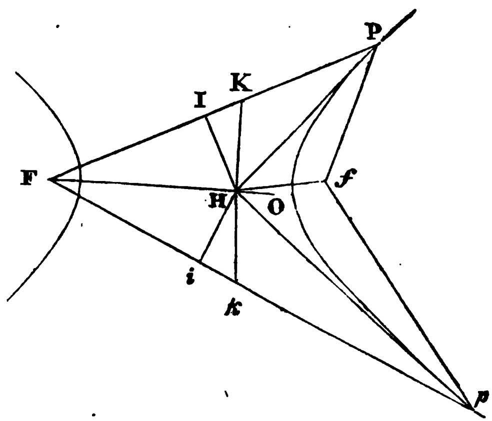
Let Pp be a diameter, HP, hp tangents at its vertices; draw straight lines from P and p to F and f, the foci. The triangles FCP, fCp, having FC = fC, CP = Cp (2), and the angles at C equal, are in all respects equal; and because the angle FPC is equal to Cpf, FP is parallel to fp (27, 1, E.), therefore Pf is equal and parallel to pF (33, 1, E.): thus FPfp is a parallelogram of which the opposite angles P and p are equal (34, 1, E.). Now the angles FPH, fph are the halves of these angles (4), therefore the angles FPH, fph, and hence CPH, Cph, are also equal, and consequently HP is parallel to hp.

Cor. 1. If tangents be drawn to an hyperbola at the vertices of a transverse diameter, straight lines drawn from either focus to the points of contact make equal angles with these tangents; for the angle Fph is equal to FPH.

Cor. 2. The transverse axis is the only diameter which is perpendicular to tangents at its vertices. For let Pp be any other diameter. The angle CPH is less than FPH, that is, less than the half of FPf, therefore CPH is less than a right angle.

PROPOSITION VII.

If a straight line be drawn from either focus of an hyperbola to the intersection of two tangents to the curve; it will make equal angles with straight lines drawn from the same focus to the points of contact.



Let HP, Hp, tangents to an hyperbola at P and p, intersect each other at H; draw PF, pF, HF to F, either of the foci; the line HF makes equal angles with FP, Fp.

Draw Pf, pf, Hf, to f, the other focus, and in FP, Fp take PK = Pf and pk = pf; join HK, Hk.

The triangles HPK, HPf have PK = Pf, PH common to both, and the angles KPH, fPH equal (5); therefore they are in every way equal, and have HK = Hf. In the same way it may be shown that the triangles Hpk, Hpf are in every way equal, and therefore that Hk = Hf.

The triangles HFK, HFk have HK = Hk (for each is equal to Hf), HF common to both, and FK = Fk, because

each is equal to PF—Pf or pF—pf, that is, to the transverse axis; therefore they are in all respects equal; and the angle HFK is equal to the angle HFk; wherefore HF makes equal angles with FP and Fp.

Cor. Perpendiculars drawn from the intersection of two tangents to straight lines drawn from either focus through the points of contact are equal. Let HI, Hi be perpendiculars drawn from H, the intersection of the tangents PH, pH on the lines FP, Fp. The triangles HFI, HFi are in all respects equal (26, 1, E.); therefore HI = Hi.

PROPOSITION VIII.

Straight lines drawn from the intersection of two tangents to the foci, make equal angles with the tangents.

Fig. 1.

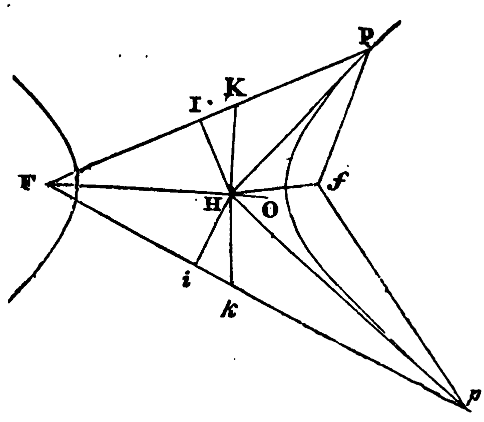
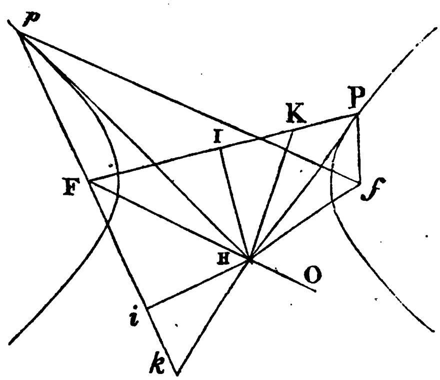


Fig. 2.



Let F, f be the foci of an hyperbola, and let straight lines HP, Hp, which intersect each other at H, touch the hyperbola at P and p; also let HF, Hf be lines drawn to the foci; and let FH be produced to any distance O, the angles PHO, pHf are equal.

The same construction being made as in Prop. 7, because the angles FHK, FHk are equal, the angles KHO, kHO are equal.

$$\text{Now KHO}\begin{cases} = \text{KH}f + f\text{HO}, \\ = 2\text{PH}f + f\text{HO}, \\ = 2\text{PHO} - f\text{HO}, \end{cases}$$

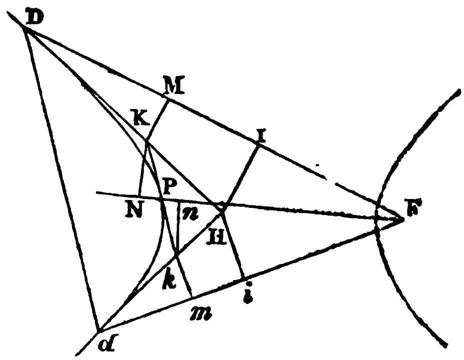
$$\text{and } k\text{HO}\begin{cases} = k\text{H}f - f\text{HO}, \\ = 2p\text{H}f - f\text{HO}; \end{cases}$$

therefore $2\text{PHO} = 2p\text{H}f$, and $\text{PHO} = p\text{H}f$.

When the tangents are drawn to opposite hyperbolas (fig. 2), the demonstration is the same as the above, except that the angle kHf being equal to twice the supplement of pHf, PHO = Supp. pHf, and consequently PHF = pHf.

PROPOSITION IX.

If two tangents to an hyperbola be at the extremities of a chord, and a third tangent be parallel to the chord, the part of this tangent intercepted by the other two is bisected at the point of contact.



Let HD, Hd be tangents at the extremities of the chord Dd; and let Kk, a tangent parallel to Dd, meet the other two tangents in K and k. The intercepted segment Kk is bisected at P, the point of contact.

From the points of contact D, P, d, draw lines to F, either of the foci; and from H, K, k, the intersections of the tangents, draw perpendiculars to the lines drawn from the points of contact to the focus, viz. HI, Hi perpendicular to DF, dF, and KM, KN perpendicular to FD, FP, and km, kn perpendicular to Fd, FP.

The triangles DHI, DKM are manifestly equiangular, also the triangles dHi, dkm;

$$\text{therefore } DH : DK = HI : KM \ (4, 6, E.),$$

$$\text{and } dH : dk = Hi : km.$$

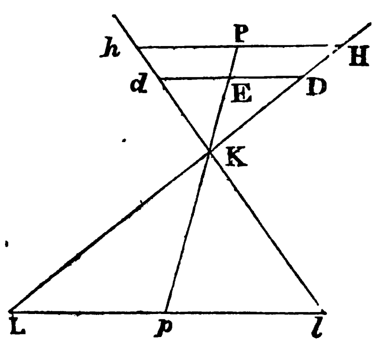
But because Dd is parallel to Kk, a side of the triangle HKk, DH : DK = dH : dk (2, 6, E.),

therefore HI : KM = Hi, km.

Now, HI = Hi (Cor. 7), therefore KM = km; but KM = KN, and km = kn (Cor. 7), therefore KN = kn; and since from the similar triangles KPN, kPn, KN : kn = KP : kP, therefore KP is equal to kP.

LEMMA III.

Let KLl be a triangle, having its base Ll bisected at p, and let Hh, any straight line parallel to the base, and terminated by the sides produced, be bisected at P, then P, p, the points of bisection, and K, the vertex of the triangle, are in the same straight line, and that line bisects Dd, any other line parallel to the base.



Join KP, Kp; the triangles KHh, KLl being similar, and Hh, Ll similarly divided at P, p,

KH : KL = (Hh : Ll) HP : Lp.

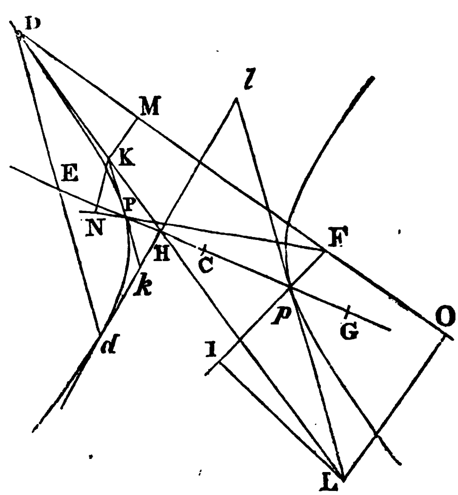
Now the angles at H and L are equal, therefore the tri-

angles KHP, KLp are similar, and the angle PKH is equal to pKL; to each add the angle HKp, and the angles PKH, HKp are equal to pKL, HKp, that is, to two right angles; therefore KP, Kp are in the same straight line (14, 1, E.)

Next let Dd meet KP in E, then
HP : DE (=PK : EK)=Ph : Ed,
But HP = Ph, therefore DE = Ed.

PROPOSITION X.

Any chord parallel to a tangent is bisected by the diameter which passes through the point of contact; or it is an ordinate to that diameter.



The chord DEd, which is parallel to Kk, a tangent at P, is bisected at E by the diameter PCp.

Draw L*p*, a tangent at *p*, the other end of the diameter, and DH, *d*H tangents at D and *d*, the extremities of the chord, meeting the other tangents in K, *k* and L, *l*. Then KP*k* and L*pl* are bisected at P and *p* (9), therefore the diameter P*p*, when produced, will pass through H, and bisect D*d*, which is parallel to K*k* or L*l* in E. (Lemma 3.)

Cor. 1. Straight lines which touch an hyperbola in the extremities of an ordinate to any diameter, intersect each other in that diameter.

Cor. 2. Every ordinate to a diameter is parallel to a tangent at its vertex; for if not, let a tangent be drawn parallel to the ordinate; then the diameter drawn through the point of contact would bisect the ordinate, and thus the same line would be bisected in two different points, which is absurd.

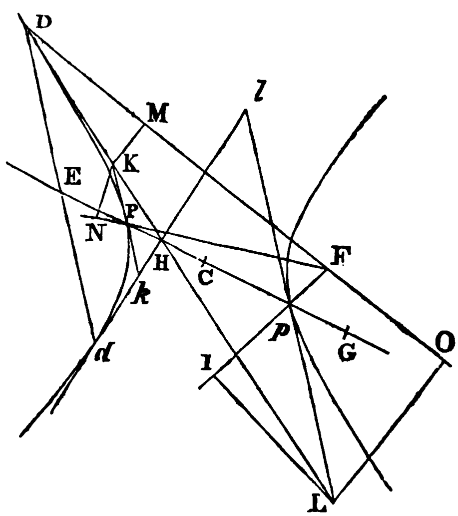
Cor. 3. All the ordinates to the same diameter are parallel to each other.

Cor. 4. A straight line that bisects two parallel chords, and terminates in the curve, is a diameter.

Cor. 5. The ordinates to either axis are perpendicular to that axis, and no other diameter is perpendicular to its ordinates.

PROPOSITION XI.

If a tangent to an hyperbola meet a transverse diameter, and from the point of contact an ordinate be drawn to that diameter, the semi-diameter will be a mean proportional between the segments of the diameter, intercepted between the centre and the ordinate, and between the centre and the tangent.



Let DH, a tangent to the hyperbola at D, meet a transverse diameter P*p* in H, and let DE*d* be an ordinate to that diameter, then CE : CP = CP : CH.

Through P and *p*, the vertices of the diameter, draw the tangents PK, *p*L, meeting DH in K and L; draw PF, *p*F to either of the foci, join DF, and draw KM and KN perpendicular to FD and FP, and also LO and LI perpendicular to FD and F*p*.

The triangles PKN, pLI are equiangular, for the angles at N and I are right angles, and the angles NPK, IpL are equal (1 Cor. 6); therefore

$$PK : pL = KN : LI \text{ (4, 6, E.)} = KM : LO : \text{(Cor. 7.)}$$

But the triangles KDM, LDO being manifestly equiangular, $KM : LO = KD : LD$;

$$\text{therefore } PK : pL = KD : LD.$$

But because of the parallel lines PK, ED, pL, the triangles HPK, HpL are equiangular, and the lines HL, Hp are similarly divided in K, D, and in P, E (10, 6, E.) hence

$$PK : pL = HP : Hp, \text{ and } KD : LD = PE : pE;$$

$$\text{therefore } HP : Hp = PE : pE.$$

Take $CG = CE$, and then $PE = pG$, and, by composition,

$$HP : Pp = PE : EG;$$

and taking the halves of the consequents,

$$HP : PC = PE : EC,$$

and, by division, $HC : PC = PC : EC$.

Cor. 1. The rectangle $PE \cdot Ep$ is equal to the rectangle $HE \cdot EC$.

For $PC^2 = HC \cdot CE$ (17, 6, E.) $= EC^2 - HE \cdot EC$ (1, 2, E.),

$$\text{also } PC^2 = EC^2 - PE \cdot Ep \text{ (6, 2, E.)};$$

$$\text{therefore } HE \cdot EC = PE \cdot Ep.$$

Cor. 2. The rectangle $PH \cdot Hp$ is equal to the rectangle $HE \cdot HC$.

$$\text{For } HC^2 = CP^2 - PH \cdot Hp \text{ (5, 2, E.)}$$

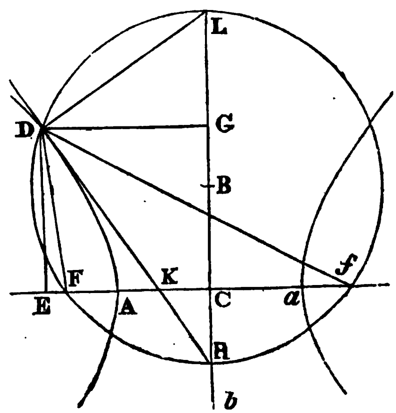
$$\text{and } HC^2 = EC \cdot HC - EH \cdot HC \text{ (1, 2, E.)} =$$

$$CP^2 - EH \cdot HC \text{ (by the Proposition)};$$

$$\text{therefore } PH \cdot Hp = EH \cdot HC.$$

PROPOSITION XII.

If a tangent to an hyperbola meet the conjugate axis, and from the point of contact a perpendicular be drawn to that axis, the semi-axis will be a mean proportional between the segments of the axis intercepted between the centre and the perpendicular, and between the centre and the tangent.



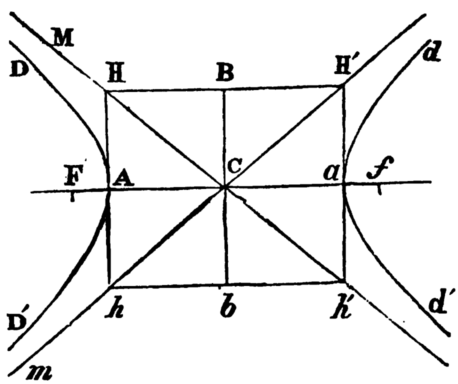
Let DH, a tangent to the hyperbola at D, meet the conjugate axis Bb in H, and let DG be perpendicular to that axis, then $\quad CG : CB = CB : CH.$

Let DH meet the transverse axis in K; draw DE perpendicular to that axis; draw DF, Df to the foci, and describe a circle about the triangle DfF; the conjugate axis will evidently pass through the centre of the circle; and because the angle FDf is bisected by the tangent DK, the line DK will pass through H, the intersection of the conjugate axis with the circumference; therefore the circle passes through H. Draw DL to the other extre-

mity of the diameter. The triangles LGD, KCH are similar, for each is similar to the right-angled triangle LDH, therefore

$$LG : GD \text{ (or } CE) = CK : CH;$$
$$\text{hence } LG \cdot CH = CE \cdot CK = CA^2 \text{ (11).}$$
$$\text{Now } LC \cdot CH = CF^2 \text{ (35, 3, E.),}$$
$$\text{therefore } LC \cdot CH - LG \cdot CH = CF^2 - CA^2;$$
$$\text{that is, } CG \cdot CH = CB^2 \text{ (Def. 8),}$$
$$\text{wherefore } CG : CB = CB : CH.$$

DEFINITION.



XII. If through A, one of the vertices of the transverse axis, a straight line HAh be drawn, equal and parallel to Bb the conjugate axis, and bisected at A by the transverse axis; the straight lines CHM, Chm drawn through the centre, and the extremities of that parallel, are called *Asymptotes.*

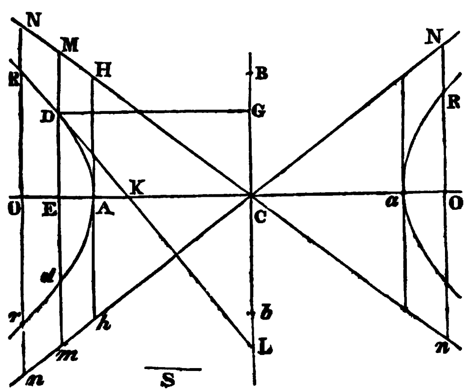
Cor. 1. The asymptotes of two opposite hyperbolas are common to both. Through a, the other extremity of the axis, draw H′ah′, parallel to Bb, and meeting the asymp-

totes of the hyperbola DAD′ in H′ and h'. Because aC is equal to AC, aH′ is equal to Ah, and ah' to AH.

Cor. 2. The asymptotes are diagonals of a rectangle formed by drawing perpendiculars to the axes at their vertices; for the lines AH, CB, aH′, being equal and parallel, the points H, B, H′, are in a straight line parallel to Aa; the same is true of the points h, b, h'.

PROPOSITION XIII.

The asymptotes do not meet the hyperbola; and if from any point in the curve a straight line be drawn parallel to the conjugate axis, and terminated by the asymptotes, the rectangle contained by its segments between that point and the asymptotes is equal to the square of half the conjugate axis.



Through D, any point in the hyperbola, draw a straight line parallel to the conjugate axis, meeting the transverse axis in E, and the asymptotes in M and m; the points

M and m are without the hyperbola, and the rectangle $MD \cdot Dm$ is equal to the square of BC.

Draw DG perpendicular to Bb the conjugate axis; let a tangent to the curve at D meet the transverse axis in K, and the conjugate axis in L, and let a perpendicular at the vertex A meet the asymptote in H. Because DK is a tangent, and DE an ordinate to the axis, CA is a mean proportional between CK and CE (11); and therefore

$$CK : CE = CA^2 : CE^2 \text{ (2 Cor. 20, 6, E.)}.$$

$$\text{But } CK : CE = LC : LG,$$

$$\text{and } CA^2 : CE^2 :: AH^2 : EM^2,$$

$$\text{therefore } LC : LG :: AH^2 : EM^2.$$

Again, CB being a mean proportional between CL and CG (12),

$$LC : CG = CB^2 : CG^2,$$

and therefore, by composition,

$$LC : LG = CB^2 : CB^2 + CG^2, \text{ or } CB^2 + ED^2;$$

$$\text{wherefore } AH^2 : EM^2 = CB^2 : CB^2 + ED^2.$$

$$\text{Now } AH^2 = CB^2 \text{ (Def. 12)},$$

$$\text{therefore } EM^2 = CB^2 + ED^2,$$

consequently EM^2 is greater than ED^2, and EM greater than ED, therefore M is without the hyperbola. In like manner it appears that m is without the hyperbola, therefore every point in both the asymptotes is without the hyperbola. Again, the straight line Mm, terminated by the asymptotes, being manifestly bisected by the axis at E,

$$ME^2 = MD \cdot Dm + DE^2 \text{ (5, 2, E.)}$$

but it has been shown that

$$ME^2 = BC^2 + DE^2,$$

$$\text{therefore } MD \cdot Dm = BC^2.$$

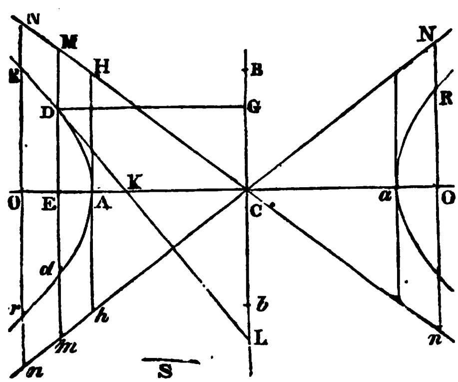
Cor. 1. Hence, if in a straight line Mm, terminated by the asymptotes, and parallel to the conjugate axis, there

be taken a point D such that the rectangle MD · D*m* is equal to the square of that axis, the point D is in the hyperbola.

Cor. 2. If straight lines MD*m*, NO*n*, be drawn through D and R, any points in the hyperbola, or opposite hyperbolas, parallel to the conjugate axis, and meeting the asymptotes in M, *m*, and N, *n*, the rectangles MD · D*m*, NR · R*n* are equal.

PROPOSITION XIV.

The hyperbola and its asymptote, when produced, continually approach to each other, and the distance between them becomes less than any given line.



Take two points E and O in the transverse axis produced, and through these points draw straight lines parallel to the conjugate axis, meeting the hyperbola in D, R, and the asymptotes in M, *m* and N, *n*.

$$\text{Because } NO^2 > ME^2$$
$$\text{and } NR \cdot Rn = MD \cdot Dm \text{ (2 Cor. 13)},$$
$$\text{therefore } NO^2 - NR \cdot Rn > ME^2 - MD \cdot Dm;$$
$$\text{that is, } RO^2 > DE^2,$$
$$\text{and } RO > DE.$$
$$\text{Now } On > Em,$$
$$\text{therefore } Rn > Dm;$$
$$\text{and since } Rn : Dm = DM : RN \text{ (2 Cor. 13)},$$
$$DM > RN,$$

therefore the point R is nearer to the asymptote than D, that is, the hyperbola, when produced, approaches to the asymptote.

Let S be any line less than half the conjugate axis; then, because Dm, a straight line drawn from a point in the hyperbola, parallel to the conjugate axis, and terminated by the asymptote on the other side of the transverse axis, may evidently be of any magnitude greater than Ah, which is equal to half the conjugate axis, Dm may be a third proportional to S and BC; and since Dm is also a third proportional to DM (the segment between D and the other asymptote) and BC, DM may be equal to S; but the distance of D from the asymptote is less than DM, therefore that distance may become less than S, and consequently less than any given line.

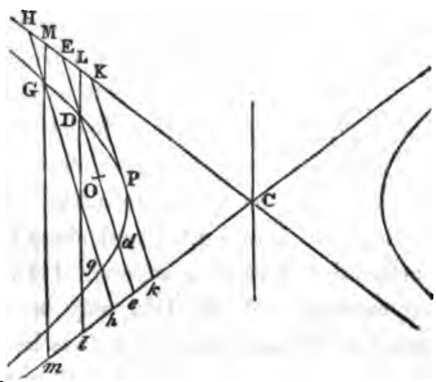
Cor. Every straight line passing through the centre within the angles contained by the asymptotes through which the transverse axis passes, meets the hyperbola, and therefore is a transverse diameter; and every straight line passing through the centre within the adjacent angles falls entirely without the hyperbola.

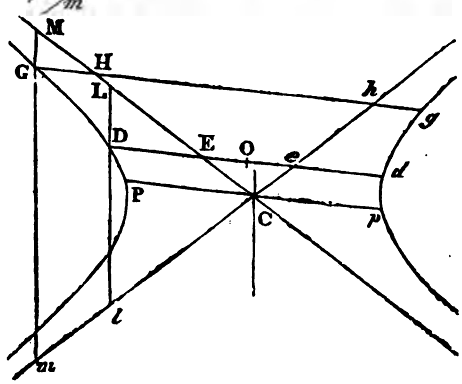
Scholium. The name *asymptotes* (*non concurrentes*) has been given to the lines CH, Ch, because of the pro-

perty they have of continually approaching to the hyperbola without meeting it, as has been proved in this proposition.

PROPOSITION XV.

If from two points in an hyperbola, or opposite hyperbolas, two parallel straight lines be drawn to meet the asymptotes, the rectangles contained by their segments between the points and the asymptotes are equal.





Let D and G be two points in the hyperbola, or opposite hyperbolas; let parallel lines EDe, HGh be drawn to meet the asymptotes in E, e, and H, h; the rectangles ED $\cdot$ De, HG $\cdot$ Gh are equal.

Through D and G draw straight lines parallel to the conjugate axis, meeting the asymptotes in the points L, l, and M, m. The triangles HGM, EDL are similar, as also the triangles hGm, eDl.

$$\text{therefore } DL : DE = GM : GH,$$
$$\text{and } Dl : De = Gm : Gh;$$

hence, taking the rectangles of the corresponding terms of the proportions,

$$LD \cdot Dl : ED \cdot De = MG \cdot Gm : HG \cdot Gh;$$
$$\text{but } LD \cdot Dl = MG \cdot Gm \text{ (2 Cor. 13)},$$
$$\text{therefore } ED \cdot De = HG \cdot Gh.$$

Cor. 1. If a straight line be drawn through D, d, two points in the same or opposite hyperbolas, the segments DE, de between those points and the asymptotes are equal. For in the same manner that the rectangles ED $\cdot$ De, GH $\cdot$ Gh have been proved to be equal, it may be shown that the rectangles Ed $\cdot$ de, HG $\cdot$ Gh are equal, therefore ED $\cdot$ De = Ed $\cdot$ de. Let Ee be bisected in O, then ED $\cdot$ De = $EO^2 - OD^2$, and Ed $\cdot$ de = $EO^2 - Od^2$, therefore $EO^2 - OD^2 = EO^2 - Od^2$; hence OD = O$d$, and ED = ed.

Cor. 2. When the points D and d are in the same hyperbola, by supposing them to approach till they coincide at P, the line Ee will become a tangent to the curve at P. Therefore any tangent KPk, which is terminated by the asymptotes, is bisected at P, the point of contact.

Cor. 3. And if any straight line KPk, limited by the asymptotes, be bisected at P, a point in the curve, that

line is a tangent at P. For it is evident that only one line can be drawn through P, which shall be limited by the asymptotes, and bisected at P.

Cor. 4. If a straight line be drawn through D, any point in the hyperbola, parallel to a tangent KPk, and terminated by the asymptotes at E and e, the rectangle ED · De is equal to the square of PK or Pk, the segment of the tangent between the point of contact and either asymptote. The demonstration is the same as in the proposition.

Cor. 5. If from any point D in an hyperbola (fig. 2) a straight line be drawn parallel to Pp, any diameter, meeting the asymptotes in E and e, the rectangle ED · De is equal to the square of half that diameter. The demonstration is the same as in the proposition.

PROPOSITION XVI.

If two straight lines be drawn from any point in an hyperbola to the asymptotes, and from any other point in the same or opposite hyperbolas two other lines parallel to the former be drawn to meet the same asymptotes; the rectangle contained by the first two lines will be equal to the rectangle contained by the other two lines.

From D, any point in the hyperbola, draw DH and DK to the asymptotes, and from any other point d draw dh and dk respectively parallel to DH and DK, and meeting the asymptotes in h and k. The rectangles HD · DK, hd · dk are equal.

Join D, d, meeting the asymptotes in E, e. From similar triangles

$$\text{ED} : \text{DH} = \text{E}d : dh,$$

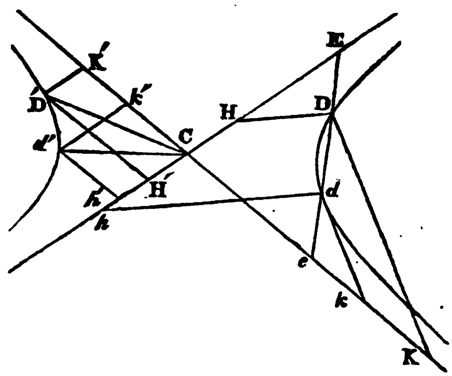
$$\text{and } e\text{D} : \text{DK} = ed : dk;$$

therefore, taking the rectangles of the corresponding terms,

$$\text{ED} \cdot \text{D}e : \text{HD} \cdot \text{DK} = \text{E}d \cdot de : hd \cdot dk;$$

$$\text{but ED} \cdot \text{D}e = \text{E}d \cdot de \text{ (1 Cor. 15)},$$

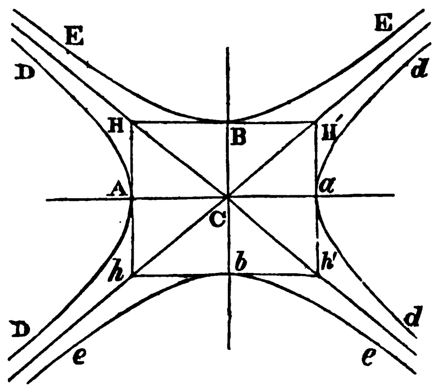
$$\text{therefore HD} \cdot \text{DK} = hd \cdot dk.$$



COR. 1. If the lines D′K′, D′H′, d′k′, d′h′, be parallel to the asymptotes, and thus form the parallelograms D′K′CH′, d′k′Ch′, these are equal to one another (14, 6, E.). And if D′C, d′C be joined, the halves of the parallelograms, or the triangles D′K′C, d′k′C are also equal.

COR. 2. If from D′, d′, any two points in an hyperbola, straight lines D′K′, d′k′ be drawn parallel to one asymptote, meeting the other in K′ and k′, these lines are to each other reciprocally as the distances of K′ and k′ from the centre, or D′K′ : d′k′ :: Ck′ : CK′. This appears from last corollary, and 14, 6, E.

DEFINITIONS.



XIII. If Aa be the transverse axis and Bb the conjugate axis of two opposite hyperbolas DAD, *dad*, and if Bb be the transverse axis and Aa the conjugate axis of other two opposite hyperbolas EBE, *ebe*, these hyperbolas are said to be *conjugate to the former*. When all the four hyperbolas are mentioned they are called *conjugate hyperbolas*.

Cor. The asymptotes of the hyperbolas DAD, *dad* are also the asymptotes of the hyperbolas EBE, *ebe*. This is evident from Cor. 2 to Definition 12.

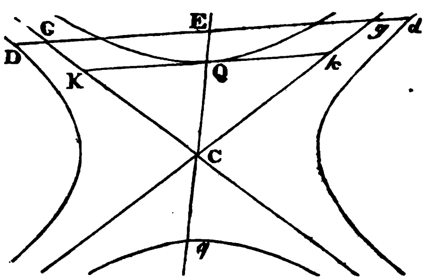
XIV. Any diameter of the conjugate hyperbolas is called a *second diameter of the other hyperbolas*.

Cor. Every straight line passing through the centre, within the angle of the asymptotes through which the conjugate axis passes, and terminated by the opposite hyperbolas, is a *second diameter* of the hyperbola.

XV. Any straight line not passing through the centre, but terminated both ways by the opposite hyperbolas, and bisected by a second diameter, is called an *ordinate to that diameter*.

PROPOSITION XVII.

Any straight line not passing through the centre, but terminated by the opposite hyperbolas, and parallel to a tangent to either of the conjugate hyperbolas, is bisected by the second diameter that passes through the point of contact, or is an ordinate to that diameter.



The straight line Dd terminated by the opposite hyperbolas, and parallel to the tangent KQk, is bisected at E by Qq, the diameter that passes through the point of contact.

Let Dd meet the asymptotes in G and g, and let the tangent meet them in K and k. The straight lines Gg, Kk are evidently similarly divided at E and Q; and since KQ = Qk (2 Cor. 15), therefore GE = Eg; now DG = gd (1 Cor. 15), therefore DE = Ed.

Cor. 1. Every ordinate to a second diameter is parallel to a tangent at its vertex. The demonstration is the same as in Cor. 2, Prop. 10.

Cor. 2. All the ordinates to the same second diameter are parallel to each other.

Cor. 3. A straight line that bisects two parallel straight

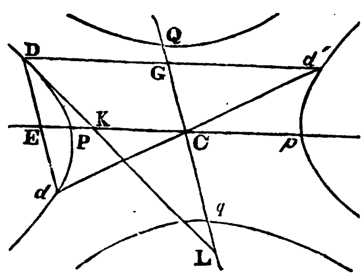
lines which terminate in the opposite hyperbolas is a second diameter.

Cor. 4. The ordinates to the conjugate or second axis are perpendicular to it, and no other second diameter is perpendicular to its ordinates.

Cor. 5. The opposite hyperbolas are similar to one another, and like portions of them are in all respects equal.

PROPOSITION XVIII.

If a transverse diameter of an hyperbola be parallel to the ordinates to a second diameter; the latter shall be parallel to the ordinates to the former.



Let P*p*, a transverse diameter of an hyperbola, be parallel to DG*d′*, any ordinate to the second diameter Q*q*; the second diameter Q*q* shall be parallel to the ordinates to the diameter P*p*.

Draw the diameter *d*C*d′* through one extremity of the ordinate *d′*D, and join *d* and D, the other extremity, meeting P*p* in E. Because *dd′* is bisected at C, and CE is parallel to *d′*D, the line D*d* is bisected at E, therefore D*d* is an ordinate to the diameter P*p*. And because D*d* and *dd′* are bisected at G and C, the diameter Q*q* is parallel to D*d* (2, 6, E.); therefore (2 Cor. 17) Q*q* is parallel to any ordinate to the diameter P*p*.

DEFINITIONS.

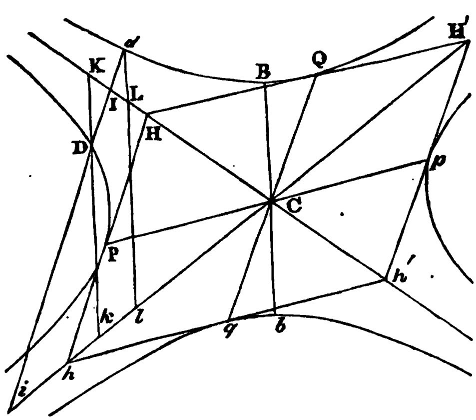
XVI. Two diameters are said to be *conjugate to one another* when each is parallel to the ordinates to the other diameter.

Cor. Diameters which are conjugate to one another are parallel to tangents at the vertices of each other.

XVII. A third proportional to any diameter and its conjugate is called the *Parameter*, also the *Latus rectum of that diameter.*

PROPOSITION XIX.

The tangent at the vertex of any transverse diameter of an hyperbola, which is terminated by the asymptotes, is equal to the diameter that is conjugate to that transverse diameter.



Let P€p be any transverse diameter of an hyperbola, HPh a tangent at its vertex, meeting the asymptotes in H and h, and Qq the diameter which is conjugate to Pp; the tangent Hh is equal to the diameter Qq.

Through D, any point in the hyperbola, draw a straight line parallel to the tangent and diameter, cutting either of the conjugate hyperbolas in d, and the asymptotes in I and i, and through D and d draw lines parallel to Bb, the conjugate axis, meeting the asymptotes in the points K, k, and L, l. The triangles DIK, dIL are similar, as also iDk, idl, therefore

$$\mathrm{KD} : \mathrm{DI} : : \mathrm{L}d : d\mathrm{I},$$

$$\text{and } k\mathrm{D} : \mathrm{D}i : : ld : di;$$

therefore, taking the rectangles of the corresponding terms,

$$\mathrm{KD} \cdot \mathrm{D}k : \mathrm{ID} \cdot \mathrm{D}i : : \mathrm{L}d \cdot dl : \mathrm{I}d \cdot di.$$

But $\mathrm{KD} \cdot \mathrm{D}k = \mathrm{BC}^2$ (13), and $\mathrm{BC}^2 = \mathrm{L}d \cdot dl$ (5 Cor. 15),

$$\text{therefore } \mathrm{ID} \cdot \mathrm{D}i = \mathrm{I}d \cdot di.$$

$$\text{Now } \mathrm{ID} \cdot \mathrm{D}i = \mathrm{HP}^2 \text{ (4 Cor. 15)},$$

$$\text{and } \mathrm{I}d \cdot di = \mathrm{QC}^2 \text{ (5 Cor. 15)};$$

$$\text{therefore } \mathrm{HP}^2 = \mathrm{QC}^2, \text{ and } \mathrm{HP} = \mathrm{QC},$$

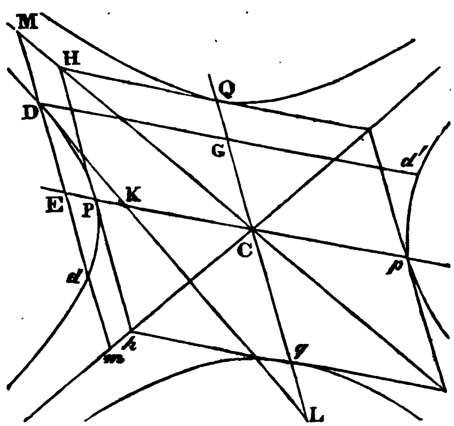
$$\text{and consequently } \mathrm{H}h = \mathrm{Q}q.$$

Cor. 1. If another tangent be drawn to the curve at p, meeting the asymptotes in H′ and h', the straight lines which join the points H, H′, also h, h', are tangents to the conjugate hyperbolas at Q and q. For pH′, as well as PH, is equal and parallel to CQ; therefore the points H, Q, H′, are in a straight line parallel to Pp, and HQ = H′Q (34, 1, E.), therefore (3 Cor. 15) HQH′ is a tangent to the curve at Q. In like manner it appears that hqh' is a tangent at q.

Cor. 2. If tangents be drawn at the vertices of two conjugate diameters, they will meet in the asymptotes, and form a parallelogram, of which the asymptotes are diagonals.

PROPOSITION XX.

If a tangent to an hyperbola meet a second diameter, and from the point of contact an ordinate be drawn to that diameter; half the second diameter will be a mean proportional between the segments of the diameter, intercepted between the centre and the ordinate, and between the centre and the tangent.



Let DL, a tangent to the curve at D, meet the second diameter Q*q* in L, and let DG*d'* be an ordinate to that diameter; then

$$CG : CQ = CQ : CL.$$

Let P*p* be the diameter that is conjugate to Q*q*; let HP*h* be a tangent at the vertex, terminated by the asymptotes; through D draw the ordinate DE*d* to the diameter P*p*, meeting the asymptotes in M and *m*; let K be the intersection of DL and P*p*. Because DK is a tangent at D, and DE*d* an ordinate to P*p*, CP is a mean proportional between CE and CK (11), and therefore

$$CE^2 : CP^2 :: CE : CK.$$

Now, the lines CQ, PH, EM, being parallel (2 Cor. 10, and Def. 16), from similar triangles,

$$CE^2 : CP^2 = EM^2 : PH^2,$$

and CE or DG : CK = LG : LC;

therefore $EM^2 : PH^2 = LG : LC$,

and, by division, &c.

$$EM^2 - PH^2 : PH^2 = CG : LC = CG^2 : CG \cdot LC.$$

But since $PH^2 = MD \cdot Dm$ (4 Cor. 15),

$$EM^2 - PH^2 = ED^2 = CG^2 \text{ (5, 2, E.)};$$

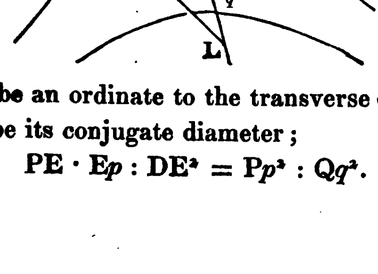
therefore $PH^2 = CG \cdot LC$;

hence, and since PH = CQ (19),

$$CG : CQ = CQ : CL.$$

PROPOSITION XXI.

If an ordinate be drawn to any transverse diameter of an hyperbola; the rectangle under the abscisses of the diameter will be to the square of the semi-ordinate as the square of the diameter to the square of its conjugate.



Let DE*d* be an ordinate to the transverse diameter P*p*, and let Q*q* be its conjugate diameter;

$$PE \cdot Ep : DE^2 = Pp^2 : Qq^2.$$

Let DKL, a tangent at D, meet the diameter in K, and its conjugate in L. Draw DG parallel to Pp, meeting Qq in G. Because CP is a mean proportional between CE and CK (11),

$$CP^2 : CE^2 = CK : CE,$$

and, by division, $CP^2 : PE \cdot Ep = CK : KE.$

But, because ED is parallel to CL,

$$CK : KE = CL : DE, \text{ or } CG;$$

and because CQ is a mean proportional between CG and CL (20),

$$CL : CG = CQ^2 : CG^2, \text{ or } DE^2,$$

therefore $CP^2 : PE \cdot Ep = CQ^2 : DE^2,$

and, by inversion and alternation,

$$PE \cdot Ep : DE^2 = CP^2 : CQ^2 = Pp^2 : Qq^2.$$

COR. 1. If an ordinate be drawn to any second diameter of an hyperbola, the sum of the squares of half the second diameter and its segment, intercepted between the ordinate and the centre, is to the square of the semi-ordinate as the square of the second diameter to the square of its conjugate.

Let DG be a semi-ordinate to the second diameter Qq. It has been shown that

$$CG^2 : CQ^2 = PE \cdot Ep : CP^2;$$

therefore, by composition,

$$CQ^2 + CG^2 : CQ^2 = CE^2 \text{ or } DG^2 : CP^2,$$

and, by alternation,

$$CQ^2 + CG^2 : DG^2 = CQ^2 : CP^2 = Qq^2 : Pp^2.$$

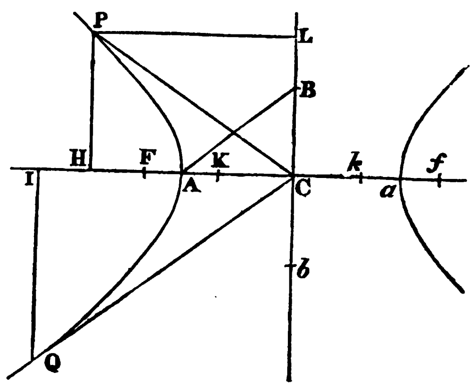
COR. 2. The squares of semi-ordinates, and of ordinates to any transverse diameter, are to one another as the rectangles contained by the corresponding abscisses; and the squares of semi-ordinates, and of ordinates to any second diameter, are to one another as the sums of the squares

of half that diameter, and the segments intercepted between the ordinate and the centre.

COR. 3. The ordinates to any transverse diameter, which intercept equal segments of that diameter from the centre, are equal to one another; and, conversely, equal ordinates intercept equal segments of the diameter from the centre.

PROPOSITION XXII.

In Aa, *the transverse axis of an hyperbola, let there be taken on each side of the centre* C, *straight lines* CK, C*k*, *each a fourth proportional to* CF *the eccentricity, and* CA, CB, *half the transverse and conjugate axes: if then from* P, *a vertex of any diameter, there be drawn* PH *perpendicular to* Aa; *the square of the semi-diameter* PC *will have to the rectangle contained by the segments* KH, *k*H *the constant ratio of the square of* CF *to the square of* CA.



Draw PL perpendicular to the conjugate axis.

Because $CB^2 : CA^2 = CB^2 + CL^2 : PL^2$ (1. Cor. 21),

by composition,

$$CB^2 + CA^2 : CA^2 = CB^2 + CL^2 + PL^2 : PL^2,$$

therefore (47, 1, E. and Def. 8)

$$CF^2 : CA^2 = CB^2 + PC^2 : PL^2 \text{ or } CH^2;$$

but, by hypothesis, $CF^2 : CA^2 = CB^2 : CK^2$,
therefore $CB^2 : CK^2 = CB^2 + PC^2 : CH^2$:
and hence (19, 5, E.)

$$PC^2 : CH^2 - CK^2 = CB^2 : CK^2, \text{ and } (6, 2, E.)$$

$$PC^2 : KH \cdot Hk = CF^2 : CA^2.$$

Cor. 1. Hence the squares of any semi-diameters PC, QC are to one another as the rectangles $KH \cdot Hk$, $KI \cdot Ik$, contained by the segments of the line Kk between its extremities, and perpendiculars from the vertices of the diameters.

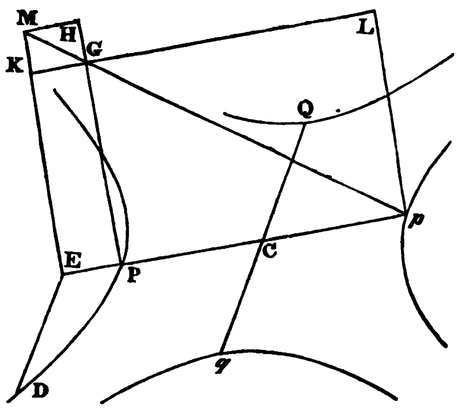
Cor. 2. The transverse axis is the least of all the transverse diameters, and a diameter which is nearer to the transverse axis is less than one more remote, and a semi-diameter may be found greater than any given line.

By hypothesis, CF or AB : CA = CB : CK. Now AB is greater than CB (19, 1, E.), therefore CA is greater than CK, and the points K, k are between Aa, the vertices of the transverse axis. Suppose now a semi-diameter PC to turn about C, and that in every position PH is perpendicular to Kk produced both ways; the rectangle $KH \cdot Hk$, and the square of PC, to which the rectangle has a constant ratio, will manifestly be least when PC coincides with AC, and both will increase as H recedes from C; and, as the rectangle may exceed any given space, the semi-diameter may become greater than any given line.

Cor. 3. Diameters which make equal angles with the transverse axis on opposite sides of it are equal; and only two equal diameters can be drawn, one on each side of the transverse axis.

PROPOSITION XXIII.

If an ordinate be drawn to any transverse diameter of an hyperbola, the rectangle under the abscisses of the diameter is to the square of the semi-ordinate as the diameter to its parameter.



Let DE be a semi-ordinate to the transverse diameter Pp; let PG be the parameter of the diameter, and Qq the conjugate diameter. By the definition of the parameter (Def. 16), $\quad \mathrm{P}p : \mathrm{Q}q = \mathrm{Q}q : \mathrm{PG}$,

$$\text{therefore } \mathrm{P}p : \mathrm{PG} = \mathrm{P}p^2 : \mathrm{Q}q^2 \text{ (2 Cor. 20, 6, E.).}$$

$$\text{But } \mathrm{P}p^2 : \mathrm{Q}q^2 = \mathrm{PE} \cdot \mathrm{E}p : \mathrm{DE}^2 \text{ (21);}$$

$$\text{therefore } \mathrm{PE} \cdot \mathrm{E}p : \mathrm{DE}^2 = \mathrm{P}p : \mathrm{PG}.$$

Cor. Let the parameter PG be perpendicular to the diameter Pp; join pG, and from E draw EM parallel to PG, meeting pG in M. The square of DE, the semi-ordinate, is equal to the rectangle contained by PE and EM.

$$\text{For } \mathrm{PE} \cdot \mathrm{E}p : \mathrm{DE}^2 = \mathrm{P}p : \mathrm{PG},$$

$$\text{and } \mathrm{P}p : \mathrm{PG} = \mathrm{E}p : \mathrm{EM} = \mathrm{PE} \cdot \mathrm{E}p : \mathrm{PE} \cdot \mathrm{EM},$$

$$\text{therefore } \mathrm{DE}^2 = \mathrm{PE} \cdot \mathrm{EM}.$$

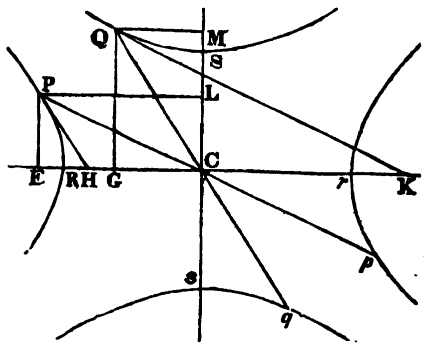
SCHOLIUM.

If the rectangles PGLp, HGKM be completed, it will appear that the square of ED is equal to the rectangle MP, which rectangle is greater than the rectangle KP, contained by the absciss PE and the parameter GP, by a rectangle KH similar and similarly situated to LP, the rectangle contained by the parameter and diameter. It was on account of the excess of the square of the ordinate above the rectangle contained by the absciss and parameter that Apollonius gave the curve to which the property belonged the name of Hyperbola.

PROPOSITION XXIV.

If from the vertices of two conjugate diameters of an hyperbola there be drawn ordinates to any third transverse diameter; the square of the segment of that diameter, intercepted between the ordinate from the vertex of the second diameter and the centre, is equal to the rectangle contained by the segments between the other ordinate and the vertices of the third transverse diameter. And the square of the segment intercepted between the ordinate from the vertex of the transverse diameter and the centre, is equal to the square of the segment between the other ordinate and the centre, together with the square of half the third transverse diameter.

Let Pp, Qq be two conjugate diameters, of which Pp is a transverse and Qq a second diameter; let PE, QG be semi-ordinates to any third transverse diameter Rr; then $CG^2 = RE \cdot Er$, and $CE^2 = CG^2 + CR^2$.



Draw the tangents PH, QK, meeting Rr in H and K. The rectangles $HC \cdot CE$ and $KC \cdot CG$ are equal, for each is equal to CR^2 (11 and 20), therefore

$$HC : CK = CG : CE.$$

But the triangles HPC, CQK are evidently similar (Cor. Def. 16); and since PE, QG are parallel, their bases CH, KC are similarly divided at E and G, therefore

$$HC : CK = HE : CG,$$

wherefore $CG : CE = HE : CG$,

consequently $CG^2 = CE \cdot EH =$ (1 Cor. 11), $RE \cdot Er$.

Again, from the similar triangles HPC, CQK,

$$HC : CK = CE : KG.$$

Now, it was shown that $HC : CK = CG : CE$,

therefore $CG : CE = CE : KG$,

consequently

$$CE^2 = CG \cdot GK = (3, 2, E.), CG^2 + GC \cdot CK;$$

$$\text{but } GC \cdot CK = CR^2 \ (20)$$
$$\text{therefore } CE^2 = CG^2 + CR^2.$$

COR. 1. Let Ss be the diameter that is conjugate to Rr, then Rr is to Ss as CG to PE, or as CE to QG.

$$\text{For } Rr^2 : Ss^2 = RE \cdot Er, \text{ or } CG^2 : PE^2 \ (21),$$
$$\text{therefore } Rr : Ss = CG : PE.$$

In like manner $Rr : Ss = CE : QG$.

COR. 2. The difference between the squares of CE, CG, the segments of the transverse diameter to which the semi-ordinates PE, QG are drawn, is equal to the square of CR the semi-diameter. For it has been shown that

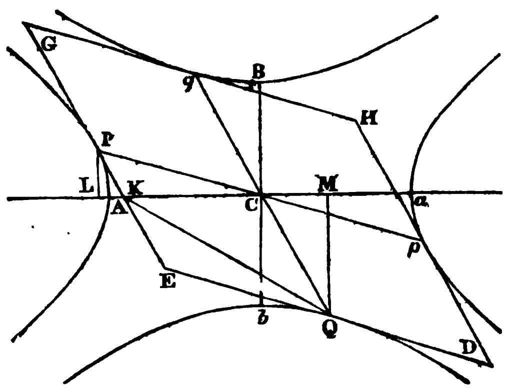
$$CE^2 = CG^2 + CR^2,$$
$$\text{therefore } CE^2 - CG^2 = CR^2.$$

COR. 3. The difference of the squares of any two conjugate diameters is equal to the difference of the squares of the axes. Let Rr, Ss be the axes, and Pp, Qq any two conjugate diameters; draw PE, QG perpendicular to Rr, and PL, QM perpendicular to Ss. Then

$$CE^2 - CG^2 = CR^2,$$
$$\text{and } CM^2 - CL^2, \text{ or } GQ^2 - PE^2 = CS^2;$$
$$\text{therefore } CE^2 + PE^2 - (CG^2 + GQ^2) = CR^2 - CS^2;$$
$$\text{that is (47, 1, E.), } CP^2 - CQ^2 = CR^2 - CS^2,$$
$$\text{therefore } Pp^2 - Qq^2 = Rr^2 - Ss^2.$$

PROPOSITION XXV.

If four straight lines be drawn touching conjugate hyperbolas at the vertices of any two conjugate diameters; the parallelogram formed by these lines is equal to the rectangle contained by the transverse and conjugate axes.



Let Pp, Qq be any two conjugate diameters, a parallelogram DEGH formed by tangents to the conjugate hyperbolas at their vertices is equal to the rectangle contained by Aa, Bb, the two axes.

Let Aa, one of the axes, meet the tangent PE in K; join QK, and draw PL, QM perpendicular to Aa.

Because CK : CA = CA : CL (11),

and CA : CB = CL : QM (1 Cor. 25),

ex. æq. CK : CB = CA : QM,

therefore CK · QM = CB · CA.

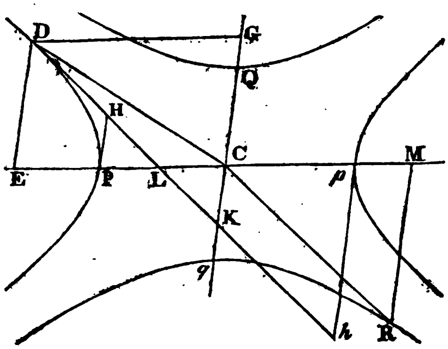
But CK · QM = twice trian. CKQ = paral. CPEQ (41, 1, E.),

therefore the parallelogram CPEQ = CB · CA;

and, taking the quadruples of these, the parallelogram DEGH is equal to the rectangle contained by Aa and Bb.

PROPOSITION XXVI.

If two tangents at the vertices of a transverse diameter of an hyperbola meet a third tangent; the rectangle contained by their segments between the points of contact and the points of intersection is equal to the square of the semi-diameter to which they are parallel: and the rectangle contained by the segments of the third tangent between its point of contact and the parallel tangents, is equal to the square of the semi-diameter to which it is parallel.



Let PH, *ph*, tangents at the vertices of a transverse diameter P*p*, meet DH*h*, a tangent to the curve at any point D, in H and *h*: let CQ be the semi-diameter to which the tangents PH, *ph* are parallel, and CR that to which DH*h* is parallel; then

$$PH \cdot ph = CQ^2 \text{ and } DH \cdot Dh = CR^2,$$

Let H*h* meet the semi-diameters CP, CQ in L and K. Draw DE, RM parallel to CQ, and DG parallel to CP.

Because $LP \cdot Lp = LE \cdot LC$ (2 Cor. 11),

$$LP : LE = LC : Lp;$$

hence, and because of the parallels PH, ED, CK, ph,

$$PH : ED = CK : ph;$$

$$\text{wherefore } PH \cdot ph = ED \cdot CK.$$

$$\text{But } ED \cdot CK = CG \cdot CK = CQ^2 \ (20),$$

$$\text{therefore } PH \cdot ph = CQ^2.$$

Again, the triangles LED, CMR are evidently similar, and LD, LE are similarly divided at H and P, also at h and p;

$$\text{therefore } PE : HD = (LE : LD =)\ CM : CR,$$

$$\text{also } pE : hD = (LE : LD =)\ CM : CR;$$

hence, taking the rectangles of the corresponding terms,

$$PE \cdot pE : HD \cdot hD = CM^2 : CR^2.$$

But if CD be joined, the points D and R are evidently the vertices of two conjugate diameters (Cor. Def. 16), and therefore $PE \cdot pE = CM^2$ (24);

$$\text{therefore } HD \cdot hD = CR^2.$$

Cor. The rectangle contained by LD and DK, the segments of a tangent intercepted between D the point of contact, and Pp, Qq, any two conjugate diameters, is equal to the square of CR, the semi-diameter to which the tangent is parallel.

Let the parallel tangents PH, ph meet LK in H and h, and draw DE, a semi-ordinate to Pp. Because of the parallels ED, PH, CK, ph,

$$LE : LD = EP : DH,$$

$$\text{and } EC : DK = Ep : Dh,$$

therefore

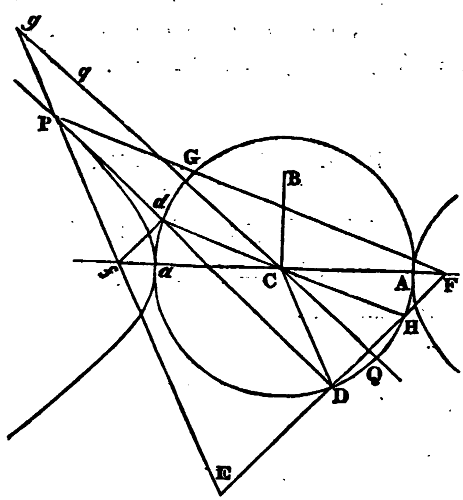
$$LE \cdot EC : LD \cdot DK = EP \cdot Ep : DH \cdot Dh.$$

$$\text{But } LE \cdot EC = EP \cdot Ep \text{ (1 Cor. 11)},$$

$$\text{therefore } LD \cdot DK = DH \cdot Dh = \text{(by this Prop.) } CR^2.$$

PROPOSITION XXVII.

If two straight lines be drawn from the foci of an hyperbola perpendicular to a tangent; straight lines drawn from the centre, to the points in which they meet the tangent, will each be equal to half the transverse axis.



Let P*d*D be a tangent to the curve at P, and FD, *fd* perpendiculars to the tangent from the foci; the straight lines joining the points C, D and C, *d* are each equal to AC, half the transverse axis.

Join FP, *f*P, and produce FD, P*f* till they intersect in E. The triangles FDP, EDP have the angles at D right angles, and the angles FPD, EPD equal (5), and the side DP common to both; they are therefore equal, and consequently have ED = DF, and EP = PF; where-

fore Ef = FP — Pf = Aa. Now the straight lines FE, Ff being bisected at D and C, the line DC is parallel to Ef, and thus the triangles FfE, FCD, are similar,

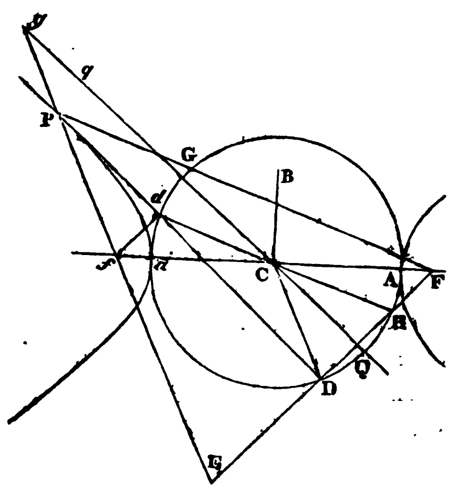
therefore Ff : fE, or Aa = FC : CD;

but FC is half Ff, therefore CD is half of Aa.

Cor. If a straight line Qq be drawn through the centre parallel to the tangent Dd, it will cut off from PF, Pf the segments PG, Pg, each equal to AC half the transverse axis. For CdPG, CDPg are parallelograms, therefore PG = dC = AC, and Pg = DC = AC.

PROPOSITION XXVIII.

The rectangle contained by perpendiculars drawn from the foci of an hyperbola to a tangent, is equal to the square of half the conjugate axis.



Let PdD be a tangent, and FD, fd perpendiculars from the foci, the rectangle contained by FD and fd is equal to the square of BC half the conjugate axis.

It is evident from last proposition that the points D, d, are in the circumference of a circle, whose centre is the centre of the hyperbola, and radius CA half the transverse axis. Now FDd being a right angle, if dC be joined and produced, it will meet DF in H, a point in the circumference; and since FC $= f$C, and CH $=$ Cd, and the angles FCH, fCd are equal, FH is equal to fd, therefore DF $\cdot$ $df =$ DF $\cdot$ FH $=$ AF $\cdot$ aF (36, 3, E.) $=$ CB^2 (3).

Cor. If PF, Pf, be drawn from the point of contact to the foci, the square of FD is a fourth proportional to fP, FP, and CB^2. For the angles fPd, FPD are equal (5), and FDP, fdP are right angles, therefore the triangles FDP, fdP are similar, and

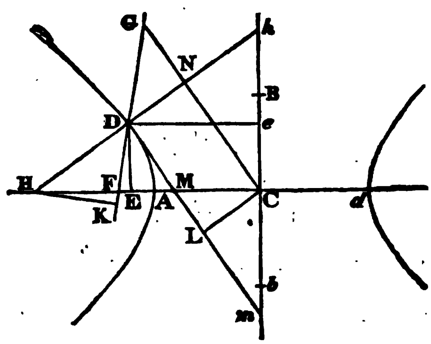
$$fP : FP = fd : FD = fd \cdot FD \text{ or } BC^2 : FD^2.$$

DEFINITION.

XVIII. A straight line perpendicular to a tangent to the curve at the point of contact is called a *Normal* to the curve.

PROPOSITION XXIX.

If from C *the centre of an hyperbola a straight line* CL *be drawn perpendicular to a tangent* LD, *and from* D *the point of contact a normal be drawn meeting the transverse axis in* H *and the conjugate axis in* h *; the rectangle contained by* CL *and* DH *is equal to the square of* CB, *the semi-conjugate axis ; and the rectangle contained by* CL *and* Dh *is equal to the square of* CA, *the semi-transverse axis.*



Let the axes meet the tangent in M and m, and from D draw the semi-ordinates DE, De, which will be perpendicular to the axes.

The triangles DEH, CLm, are evidently equiangular,

$$\text{therefore } DH : DE = Cm : CL,$$

$$\text{hence } CL \cdot DH = DE \cdot Cm\,;$$

$$\text{but } DE \cdot Cm, \text{ or } Ce \cdot Cm = BC^2 \ (12),$$

$$\text{therefore } CL \cdot DH = BC^2.$$

In the same way it may be shown that $CL \cdot Dh = AC^2$.

COR. 1. The segments DH, Dh of a normal intercepted between the point of contact and the axes are to each other reciprocally as the squares of the axes by which they are terminated.

For $AC^2 : BC^2 = CL \cdot Dh : CL \cdot DH = Dh : DH$.

COR. 2. If DF be drawn to either focus, and HK be drawn perpendicular to DF; the straight line DK shall be equal to half the parameter of the transverse axis.

Draw CG parallel to the tangent at D, meeting DH in N, and DF in G. The triangles GDN, HDK, are similar,

$$\text{therefore } GD : DN = HD : DK;$$

$$\text{and hence } GD \cdot DK = HD \cdot DN.$$

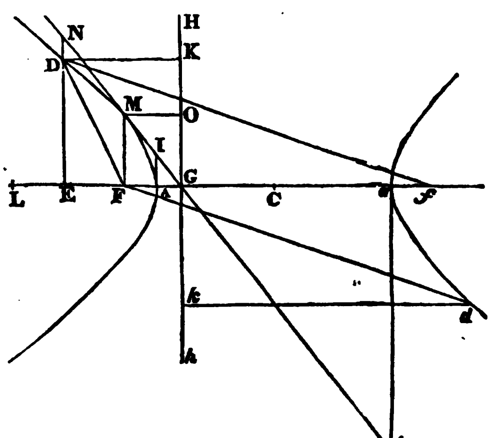
$$\text{But } GD = AC \text{ (Cor. 27) and } ND = CL,$$

$$\text{therefore } AC \cdot DK = HD \cdot CL = \text{(by the Prop.) } CB^2,$$

$$\text{wherefore } AC : BC = BC : DK;$$

hence DK is half the parameter of Aa (Def. 17).

DEFINITION.



XIX. If a point G be taken in the transverse axis of an hyperbola, so that the distance of G from the centre

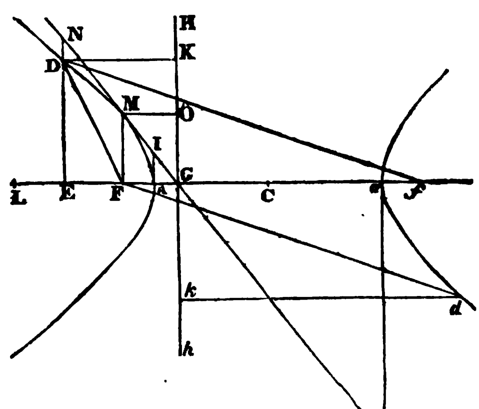
may be a third proportional to CF, the distance of either focus from the centre, and CA the semi-transverse axis; a straight line HG*h* drawn through G, perpendicular to the axis, is called the *directrix* of the hyperbola.

COR. 1. If an ordinate to the axis be drawn through the focus; tangents to the hyperbola at the extremities of the ordinate will meet the axis at the point G (11).

COR. 2. The hyperbola has two directrices, for the point G may be taken on either side of the centre.

PROPOSITION XXX.

The distance of any point in an hyperbola from either directrix is to its distance from the focus nearest that directrix, in the constant ratio of the semi-transverse axis to the distance of the focus from the centre.



Let D be any point in the hyperbola; let K be drawn perpendicular to the directrix, and DF to the focus near-

est the directrix; DK is to DF as CA, half the transverse axis, to CF, the distance of the focus from the centre.

Draw D*f* to the other focus, and DE perpendicular to A*a*; take L a point in the axis so that AL = FD, and consequently L*a* = D*f*; then CL is evidently half the sum of AL and *a*L, or of FD and *f*D, and CE half the sum of FE and *f*E: and because

$$Df - DF : Ff = fE + FE : Df + DF \text{ (K, 6, E.)},$$

by taking the halves of the terms of the proportion,

$$CA : CF = CE : CL.$$

$$\text{But } CA : CF = CG : CA \text{ (Def. 19)},$$

$$\text{therefore } CG : CA = CE : CL;$$

$$\text{hence (19, 5, E.) } EG : AL = CG : CA = CA : CF,$$

$$\text{that is, } DK : DF = CA : CF.$$

Cor. 1. If the tangent GMN be drawn through M, the extremity of the ordinate passing through the focus, and ED be produced to meet GM in N; EN shall be equal to DF. For, draw MO perpendicular to the directrix, then, because M and D are points in the hyperbola, and from similar triangles,

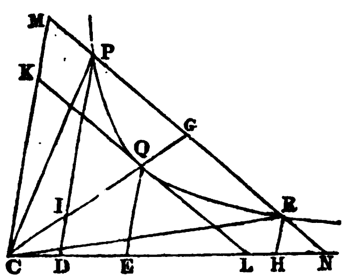
$$FM : FD = MO : DK = GF : GE = MF : EN,$$

$$\text{therefore } FD = EN.$$

Cor. 2. If AI and *ai* be drawn perpendicular to the transverse axis at its extremities, meeting the tangent GM in I and *i*, then, by the preceding corollary, AI = AF and *ai* = *a*F.

PROPOSITION XXXI.

If through P *and* Q *the vertices of two semi-diameters of an hyperbola there be drawn straight lines* PD, QE *parallel to one of the asymptotes* CM, *meeting the other asymptote in* D *and* E; *the hyperbolic sector* PCQ *is equal to the hyperbolic trapezium* PDEQ.



Let CQ meet PD in I. The triangles CDP, CEQ are equal (1 Cor. 16); therefore, taking the triangle CDI from both, the triangle CIP is equal to the quadrilateral DEQI. To these add the figure PIQ, and the hyperbolic sector PCQ is equal to the hyperbolic trapezium PDEQ.

PROPOSITION XXXII.

If from the centre of an hyperbola the segments CD, CE, CH *be taken in continued proportion in one of the asymptotes, and the straight lines* DP, EQ, HR *be drawn parallel to the other asymptote, meeting the hyperbola in* P, Q, R; *the hyperbolic areas* PDEQ, QEHR *are equal.*

Through Q (fig. of last Prop.) draw a tangent to the curve, meeting the asymptotes in K and L: join PR, meeting the asymptotes in M and N; draw the semi-diameters CP, CQ, CR; and let CQ meet PR in G.

Because QE is parallel to CM, and KQ is equal to QL (2 Cor. 15), CE is equal to EL; and because MC, PD, RH, are parallel, and MP is equal to RN (1 Cor. 15), CD is equal to HN. Now, by hypothesis,

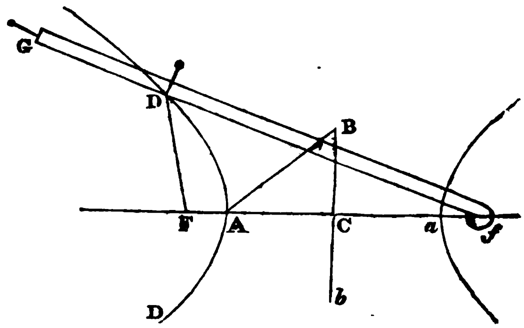
$$CD : CE = CE : CH,$$
$$\text{therefore } NH : LE = CE : CH;$$
$$\text{but } CE : CH = HR : EQ \text{ (2 Cor, 16),}$$
$$\text{therefore } NH : LE = HR : EQ,$$
$$\text{and, by alternation, } NH : HR = LE : EQ.$$

Now the angles at H and E are equal, therefore the triangles NHR, LEQ are equiangular, and NR is parallel to LQ; consequently RP is an ordinate to the diameter CQ (10), and is bisected by it at G; and as CQ bisects all lines which are parallel to KL, and are terminated by the hyperbola, it will bisect the area PQR. Let the equal areas PQG, RQG be taken from the equal triangles PCG, RCG, and there will remain the hyperbolic sectors PCQ, RCQ equal to each other. Therefore (31) the areas DPQE, EQRH are also equal.

Cor. Hence if CD, CE, CH, &c. any number of segments of the asymptote, be taken in continued proportion, the areas DPQE, DPQRH, &c. reckoned from the first line DP, will be in arithmetical progression.

PROPOSITION XXXIII. Problem.

Two straight lines Aa, Bb, *which bisect each other at right angles in* C, *being given by position; to describe an hyperbola, of which* Aa *shall be the transverse and* Bb *the conjugate axes.*



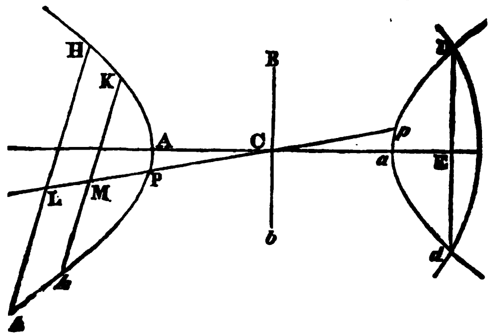
Join AB, and in Aa produced take CF, Cf each equal to AB; the points F, f will be the foci of the hyperbola. (Def. 8.)

Let one end of a string be fastened at F, and the other to G the extremity of a ruler fDG, and let the difference between the length of the ruler and the string be equal to Aa. Let the other end of the ruler be fixed to the point f, and let the ruler be made to revolve about f as a centre in the plane in which the axes are situated, while the string is stretched by means of a pin D, so that the part of it between G and D is applied close to the edge of the ruler; the point of the pin will by its motion trace a curve line DAD upon the plane, which is one of the hyperbolas required.

If the ruler be made to revolve about the other focus F, while the end of the string is fastened to f, the opposite hyperbola will be described by the moving point D; for in either case Gf — (GD + DF), that is, Df — DF is by hypothesis equal to Aa the transverse axis.

PROPOSITION XXXIV. PROBLEM.

An hyperbola being given by position; to find its axes.



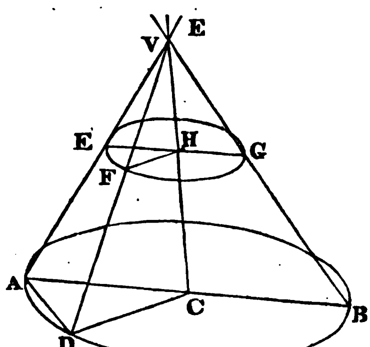
Let H*A*h be the given hyperbola. Draw two parallel straight lines Hh, Kk, terminating in either of the opposite hyperbolas, and bisect them at L and M; join LM, and produce it to meet the hyperbola in P; then LP will be a transverse diameter (4 Cor. 10). Let p be the point in which it meets the opposite hyperbola, bisect Pp in C, the point C is the centre (2). Take D any point in the hyperbola, and on C as a centre with the distance CD describe a circle; if this circle lie wholly without the opposite hyperbolas, then CD must be half the transverse axis (2 Cor. 22); but if not, let the circle meet the hyperbola again in d: join Dd, and bisect it in E; join CE, meeting the opposite hyperbolas in A and a, then Aa will be the transverse axis (5 Cor. 10); for it is perpendicular to Dd (3, 3, E.), which is an ordinate to Aa. The other axis will be found by drawing Bb a straight line through the centre perpendicular to Aa, and taking CB so that BC^2 may be a fourth proportional to the rectangle AE · Ea, and the squares of DE and CA; thus CB is half the conjugate axis (21).

PART IV.

SECTION I.

OF THE CONE AND ITS SECTIONS.

DEFINITIONS.



I. If through the point V, without the plane of the circle ABD, a straight line AVE be drawn, and produced indefinitely both ways, and if the point V remain fixed while the straight line AVE is moved round the whole circumference of the circle; two superficies will be generated by its motion, each of which is called a *Conical Superficies*, and these mentioned together are called *Opposite Conical Superficies.*

II. The solid contained by the conical superficies and the circle ADB is called a *Cone.*

III. The fixed point V is called the *Vertex of the cone.*

IV. The circle ADB is called the *Base of the cone.*

V. Any straight line drawn from the vertex to the circumference of the base is called a *Side of the cone.*

VI. A straight line VC drawn through the vertex of the cone, and the centre of the base, is called the *Axis of the cone.*

VII. If the axis of the cone be perpendicular to the base, it is called a *Right cone.*

VIII. If the axis of the cone be not perpendicular to the base, it is called a *Scalene cone.*

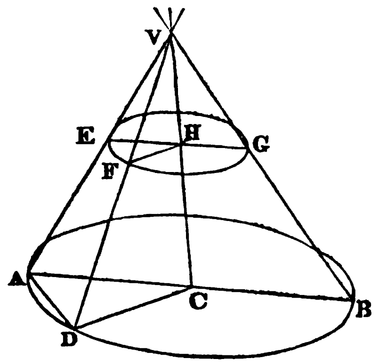
PROPOSITION I.

If a cone be cut by a plane passing through the vertex; the section will be a triangle.

Let ADBV (fig. page 128) be a cone of which VC is the axis; let AD be the common section of the base of the cone and the cutting plane; join VA, VD. When the generating line comes to the points A and D, it is evident that it will coincide with the straight lines VA, VD; these lines are therefore in the surface of the cone, and they are in the plane which passes through the points V, A, D; therefore the triangle VAD is the common section of the cone, and the plane which passes through its vertex.

PROPOSITION II.

If a cone be cut by a plane parallel to its base; the section will be a circle, the centre of which is in the axis.



Let EFG be the section made by a plane parallel to the base of the cone, and VAB, VCD two sections of the cone made by any two planes passing through the axis VC; let EG, HF be the common sections of the plane EFG, and the planes VAB, VCD. Because the planes EFG, ADB are parallel, HE, HF will be parallel to CA, CD, and

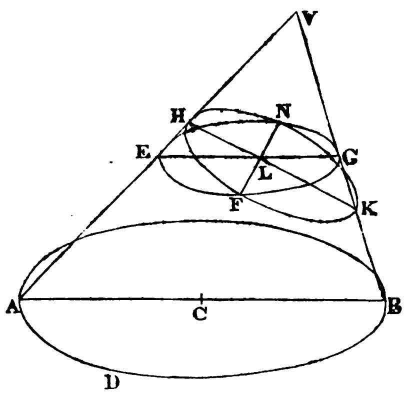
$$AC : EH = (VC : VH =)\ CD : HF;$$

but AC = CD, therefore EH = HF. For the same reason GH = HF, therefore EFG is a circle of which H is the centre and EG the diameter.

PROPOSITION III.

If a scalene cone ADBV *be cut through the axis by a plane perpendicular to the base, making the triangle* VAB, *and from any point* H *in the straight line* AV *a straight line* HK *be drawn in the plane of the triangle* VAB, *so that*

the angle VHK *may be equal to the angle* VBA, *and the cone be cut by another plane passing through* HK *perpendicular to the plane of the triangle* ABV, *the common section* HFKN *of this plane and the cone will be a circle.*



Take any point L in the straight line HK, and through L draw EG parallel to AB, and let EFGN be a section parallel to the base, passing through EG; then the two planes HFKN, EFGN being perpendicular to the plane VAB, their common section FLN is perpendicular to ELG, and since EFGN is a circle (by last Prop.), and EG its diameter, the square of FL is equal to the rectangle contained by EL and LG (35, 3, E.); but since the angle VHK is equal to VBA or VGE, the angles EHK, EGK are equal, therefore the points E, H, G, K, are in the circumference of a circle (21, 3, E.), and HL · LK = EL · LG (35, 3, E.) = FL^2, therefore the section HFKN is a circle of which HLK is a diameter (35, 3, E.)

This particular section of the cone is called a *Subcontrary Section.*

PROPOSITION IV.

If a cone be cut by a plane which does not pass through the vertex, and which is neither parallel to the base nor to the plane of a subcontrary section; the common section of the plane and the surface of the cone will be an ellipse, a parabola, or an hyperbola, according as the plane passing through the vertex parallel to the cutting plane falls without the cone, touches it, or falls within it.

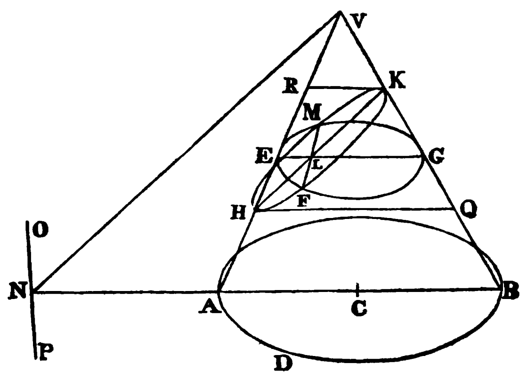
Let ADBV be any cone, and let ONP be the common section of a plane passing through its vertex and the plane of the base, which will either fall without the base, or touch it, or fall within it.

Let FKM be a section of the cone parallel to VPO; through C the centre of the base draw CN perpendicular to OP, meeting the circumference of the base in A and B; let a plane pass through V, A, and B, meeting the plane OVP in the line NV, the surface of the cone in VA, VB, and the plane of the section FKM in LK; then, because the planes OVP, MKF are parallel, KL will be parallel to VN, and will meet VB in K; it will also meet VA in H (fig. 1) within the cone; or it will be parallel to VA (fig. 2); or will meet VA, produced beyond the vertex, in H (fig. 3), according as ONP falls without the base, or touches it, or falls within it.

Let EFGM be a section of the cone parallel to the base, meeting the plane VAB in EG, and the plane FKM in FM, and let L be the intersection of EG and FM; then EG will be parallel to BN, and FM will be parallel to PO, and therefore will make the same angle with LK, wher-

ever the lines FM, LK cut each other; and since BN is perpendicular to PO, EG is perpendicular to FM. Now the section EFGM is a circle of which EG is the diameter (2), therefore FM is bisected at L, and $FL^2 = EL \cdot LG$.

Fig. 1.

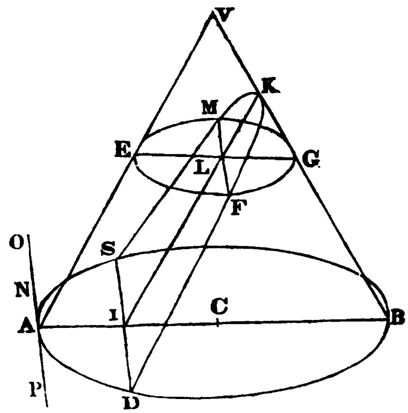


Case 1. Let the line PNO be without the base of the cone. Through K and H draw KR and HQ parallel to AB. The triangles KLG, KHQ are similar, as also HLE, HKR; therefore $KL : LG = KH : HQ$,

$$\text{and } HL : LE = KH : KR;$$

therefore $KL \cdot HL : LG \cdot LE$ or $LF^2 :: KH^2 : HQ \cdot KR$. Now the ratio of KH^2 to $HQ \cdot KR$ is the same wherever the sections HFKM, EFGM intersect each other; therefore $KL \cdot HL$ has a constant ratio to LF^2, consequently (1 Cor. 13, Part II.) the section HFKM is an ellipse, of which HK is a diameter and MF an ordinate.

Fig. 2.



CASE 2. Next, suppose the line ONP to touch the circumference of the base in A. Let DIS be the common section of the base and the plane FKM; the line DIS is evidently parallel to FLM, and perpendicular to AB, therefore $DI^2 = AI \cdot IB$,

$$\text{hence } DI^2 : FL^2 = AI \cdot IB : EL \cdot LG.$$

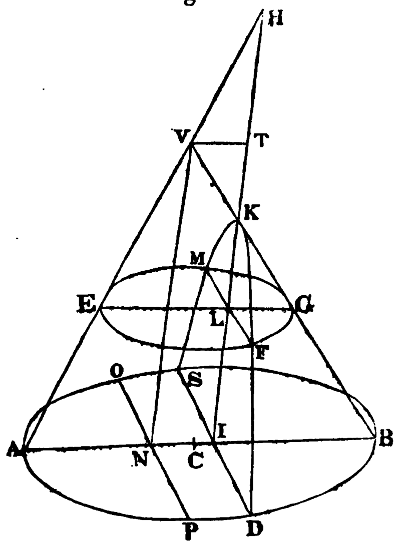
But since EG is parallel to AB, and IK parallel to AV, AI is equal to EL, and

$$IB : LG = KI : KL;$$

$$\text{therefore } DI^2 : FL^2 = KI : KL.$$

Hence it appears (Cor. 11, Part I.) that the section DFKMS is a parabola, of which KLI is a diameter, and DIS, FLM ordinates to that diameter.

Fig. 3.



CASE 3. Lastly, let the line PNO fall within the base; draw VT through the vertex parallel to EG. The triangles HVT, HEL are similar, as also the triangles KVT, KGL, therefore

$$HT : TV = HL : LE,$$
$$\text{and } KT : TV = KL : LG,$$

therefore $HT \cdot KT : TV^2 = HL \cdot LK : LE \cdot LG$ or LF^2. Hence it appears that $HL \cdot LK$ has to LF^2 a constant ratio, therefore the section DFKMS is an hyperbola of which KH is a transverse diameter, and FM an ordinate to that diameter (2 Cor. 21, Part III.)

SCHOLIUM.

From the four preceding propositions it appears, that the only lines which can be formed by the common section of a plane and the surface of a cone, are these

five: 1. A straight line, or rather two straight lines intersecting each other in the vertex of the cone, and forming with the straight line which joins the points in which they meet the base, a triangle; 2. A circle; 3. An ellipse; 4. A parabola; 5. An hyperbola. The first two of these, however, viz. the triangle and circle, may be referred to the hyperbola and the ellipse; for if the axes of an hyperbola be supposed to retain a constant ratio to each other, and, at the same time, to diminish continually, till at last the vertices coincide; the opposite hyperbolas will evidently become two straight lines intersecting each other in a point; and a circle may be considered as an ellipse, whose axes are equal, or whose foci coincide with the centre; so that the only three sections which require to be separately considered are the *ellipse*, the *parabola*, and the *hyperbola*.

SECTION II.

OF THE CURVATURE OF THE CONIC SECTIONS.

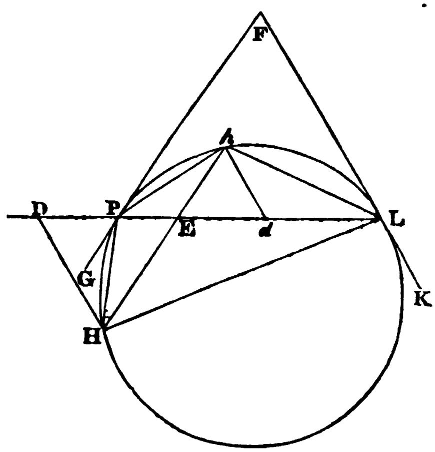
DEFINITIONS.

I. A circle is said to *touch* a conic section in any point when the circle and conic section have a common tangent at that point.

II. If a circle touch a conic section in any point, so that no other circle touching it in the same point can pass between it and the conic section on either side of the point of contact, it is said to have the *same curvature* with the conic section in the point of contact, and is called the *Circle of equal Curvature* at that point.

LEMMA.

If straight lines be drawn touching a circle at the extremities of any chord, and from any point in the circumference straight lines be drawn parallel to the tangents, to terminate in the chord; these lines will be equal; and the square of each will be equal to the rectangle contained by the segments of the chord between each line and the point of contact of the tangent to which it is parallel.



Let PL be any chord in a circle, and FPG, FLK tangents at P and L; if from any point H in the circumference there be drawn HE, HD parallel to FG, FL respectively, meeting the chord in E and D; the lines HE, HD are equal; and the square of each is equal to the rectangle LD · PE.

The sides of the triangle HDE being parallel to those of the triangle FLP, viz. HD to FL and HE to FP, the triangles are equiangular (29, 1, E.); now the angles FPL, FLP of the latter are equal (32, 3, E.), therefore the corresponding angles HED, HDE of the former are equal, and HD = HE.

And because the angle DHL is equal to HLK (29, 1, E.), and this last is equal to EPH (32, 3, E.); also, because the angle HLD is equal to HPG (32, 3, E.), which again is equal to PHE (29, 1, E.); the triangles DHL, EPH are equiangular. Therefore

$$LD : DH = HE : EP,$$

and (16, 6, E.) $LD \cdot EP = DH \cdot HE = HD^2 = HE^2$.

If the point h be in the arc of the opposite segment, and hd be drawn parallel to FL, and hE to PF, it will in like manner appear that

$$Ld \cdot EP = hd^2 = hE^2.$$

COR. 1. The points D and d being determined as directed in the proposition,

$$LP : LD = LH^2 : LP^2,$$

$$\text{and } LP : Ld = Lh^2 : Ld^2.$$

The triangles DLH, HLP have the angle at L common to both, and the angles DHL, HPL equal, because each is equal to the angle HLK (29, 1, E. and 32, 3, E.), therefore they are equiangular; hence

$$LP : LH = LH : LD \text{ (4, 6, E.)},$$

and $LP : LD\ (= LP^2 : LH^2) = LH^2 : LD^2$ (Cor. 19, 6, E.).
In the same way it may be proved that

$$LP : Ld = Lh^2 : Ld^2.$$

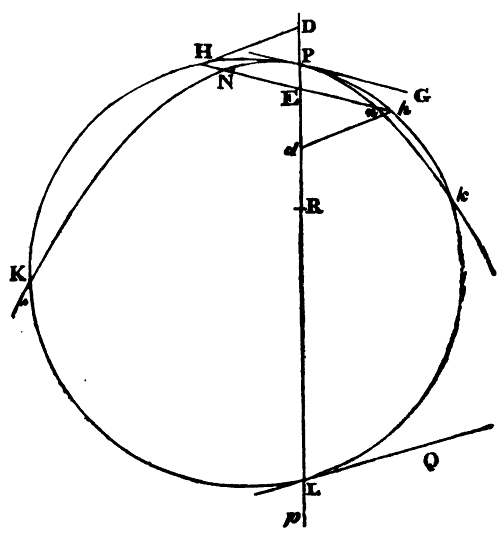
COR. 2. If E, the intersection of the chords, Hh, PL, be between P and L, the points D, d will be on opposite sides of the point P. For in this case the chord LH will be greater than the chord LP, and the chord Lh will be less; therefore LH^2 will be greater than LP^2, and Lh^2 less; consequently (from Cor. 1) LD will be greater than LP, and Ld less than LP.

COR. 3. If the chord Hh which is parallel to the tangent PF, be supposed to approach continually towards that tangent; the points D, d, and E will continually approach to P, and may come nearer to it than any assignable distance.

PROPOSITION I.

If a circle be described touching a conic section, and cutting off from the diameter that passes through the point of contact a segment greater than the parameter of that diameter; a part of the circumference on each side of the point of contact will be wholly without the conic section; but if it cut off a segment less than the parameter, a part of the circumference on each side of the point of contact will be wholly within the conic section.

Fig. 1.



CASE 1. Let the section be a parabola KNP*nk*; let P*p* be any diameter, and PG a tangent at P its vertex. Let a circle KHP*hk* touch the parabola and tangent at P; and

cut off from the diameter a segment PL either greater or less than its parameter (fig. 1, 2). An arc HP*h* of the circle, extending to each side of the vertex P, will be wholly without the parabola or wholly within it.

First let PL be greater than RL, a segment of the diameter equal to the parameter (fig. 1); draw LQ touching the circle at L; let N*n*, an ordinate to the diameter P*p*, meet the circle in H and *h*, and draw HD, *hd* parallel to LQ, meeting the diameter in D and *d*.

Because $NE^2 = nE^2 = PE \cdot LR$ (12, Part I.),

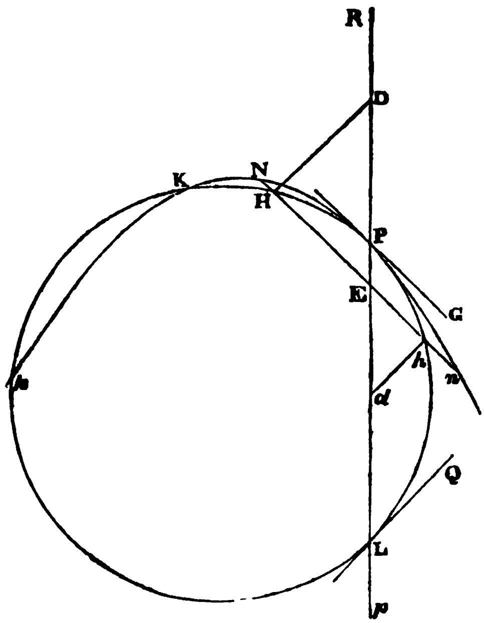
$$\left.\begin{aligned} \text{and } HE^2 &= PE \cdot LD \\ \text{also } hE^2 &= PE \cdot Ld \end{aligned}\right\} \text{(Lemma)};$$

$$\text{therefore } NE^2 : HE^2 = LR : LD,$$

$$\text{and } nE^2 : hE^2 = LR : Ld.$$

Now since P*d* may be less than any given line (Cor. 3 to Lemma), let it be less than PR; then LD and L*d* will both be greater than LR, and consequently HE will be greater than NE, and *h*E greater than *n*E; therefore the arc of the circle HP*h* will be wholly without the parabola.

Fig. 2.



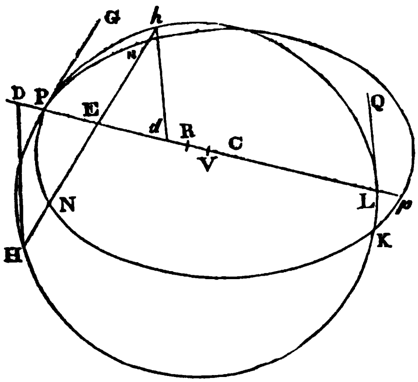
Next let PL, the segment of the diameter cut off by the circle, be less than LR the parameter; as before, let N*n*, an ordinate to the diameter P*p*, meet the circle in H and *h*, and let HD, *hd* be parallel to the tangent LQ; then, as in the preceding case,

$$NE^2 : HE^2 = LR : LD,$$

$$\text{and } nE^2 : hE^2 = LR : Ld.$$

Suppose now PD to be less than PR, then LD and L*d* will each be less than LR, and therefore HE will be less than NE, and *h*E less than *n*E; hence the arc HP*h* of the circle will be within the parabola.

Fig. 3.



CASE 2. Now let the curve be an ellipse or hyperbola (fig. 3 and 4).[1] First let the circle, which has a common tangent PG with the curve at the vertex of the diameter, cut off from it a segment PL greater than LR, a segment of the chord equal to its parameter. Let Nn, an ordinate to Pp, meet the circle in H and h: draw LQ touching the circle at L, and draw HD, hd parallel to LQ. Take a point V in Pp, such that

$$Pp : pE = LR : LV;$$

$$\text{then } Pp : LR = Ep : LV.$$

Now $PE \cdot Ep : NE^2 = Pp : LR = Ep : LV$ (15, Part II., and 21, Part III.);

hence also $PE \cdot Ep : NE^2 = PE \cdot Ep : PE \cdot LV$ (1, 6, E.),

$$\text{therefore } NE^2 = nE^2 = PE \cdot LV.$$

$$\text{Now } HE^2 = PE \cdot LD, \text{ and } hE^2 = PE \cdot Ld,$$

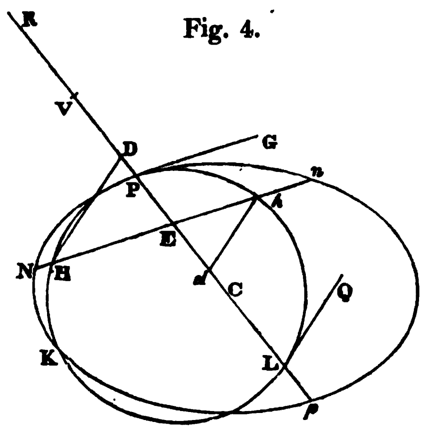
$$\text{therefore } NE^2 : HE^2 = LV : LD,$$

$$\text{and } nE^2 : hE^2 = LV : Ld.$$

[1] The reasoning in the case of the hyperbola is exactly like that for the ellipse; therefore, to avoid multiplying figures, those for the hyperbola are omitted.

Let the chord H*h* of the circle have such a position that P*d* is less than PR; then *pd* and *p*D will both be greater than *p*R, and consequently greater than *p*V, which is less than *p*R. In this position of the chord, and in every other nearer to the tangent, NE will be less than HE, and *n*E less than *h*E; therefore the arc HP*h* of the circle will be entirely without the ellipse or hyperbola on each side of their common point P.

Fig. 4.



Lastly, suppose that the segment LP of the diameter cut off by the circle is less than LR its parameter; then the same construction being made as in the other case, we shall have

$$NE^2 : HE^2 = LV : LD,$$
$$nE^2 : hE^2 = LV : Ld.$$

Now, when the ordinate N*n* approaches toward the tangent PG, the point D will approach to P, and the point V to R; therefore there will be a position of the ordinate in which LD and L*d* will be both less than LV; and the same will be true for all positions nearer to the tangent. In these, NE will be greater than HE, and *n*E greater

than hE; thus it appears that the arc HPh will be within the curve, to a certain extent, on each side of the point P.

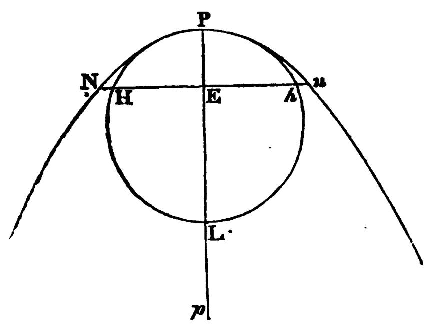
Cor. If a circle touch a conic section, and cut off from the diameter that passes through the point of contact a segment equal to its parameter, it will have the same curvature with the conic section in the point of contact.

For if a greater circle be described, it will cut off a segment greater than its parameter, therefore a part of its circumference on each side of the point of contact will be wholly without the conic section; and as it will also be without the former circle, it will not pass between that circle and the conic section at the point of contact. If, on the other hand, a less circle be described, it will cut off from the diameter a less segment than its parameter, therefore a part of the circumference on each side of the point of contact will fall within the conic section; and as it will be within the former circle, it will not pass between that circle and the conic section at the point of contact. Hence (Def. 2) the circle which cuts off a segment equal to the parameter is the circle of equal curvature.

PROPOSITION II.

The circle of curvature at the vertex of the axis of a parabola, or at the vertex of the transverse axis of an ellipse or hyperbola, falls wholly within the conic section; but the circle of curvature at the vertex of the conjugate axis of an ellipse falls wholly without the conic section.

Fig. 1.



Let Pp be the axis of a parabola, and $PHLh$ the circle of curvature at its vertex, which therefore (Cor. 1) cuts off from the axis a segment PL equal to the parameter of the axis; if a tangent were drawn to the parabola at its vertex, it would also be a tangent to the circle at that point (Def.), therefore the centre of the circle is in Pp. Let NEn, an ordinate to the axis, meet the circle in H and h. It may be shown, as in the preceding Proposition, that

$$NE^2 : HE^2 = LP : LE.$$

Now, in every position of the ordinate, LP is greater than

LE; therefore NE^2 is always greater than HE^2 and nE^2 is greater than hE^2; therefore the circle is wholly within the parabola.

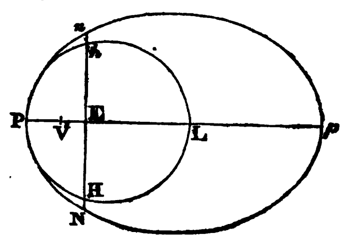
Next, let Pp be the transverse axis of an ellipse (fig. 2), or hyperbola (fig. 3), or the conjugate axis of an ellipse (fig. 4), and $PHLh$ the circle of curvature; then, as in the parabola, the centre of the circle will be in the axis. In each case draw Nn, an ordinate to the axis meeting the circle in H and h, and take a point V in PL, so that

$$pP : pE = LP : LV;$$

then it will appear, as in the last Proposition, that

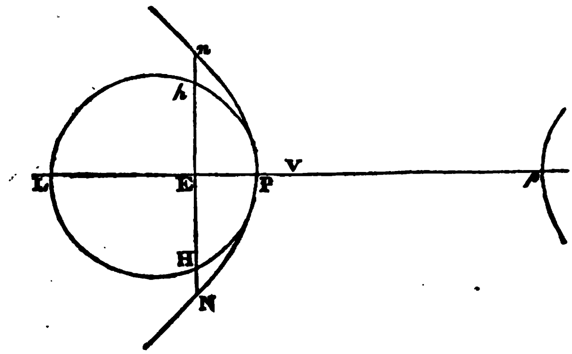
$$NE^2 : HE^2 = LV : LE.$$

Fig. 2.



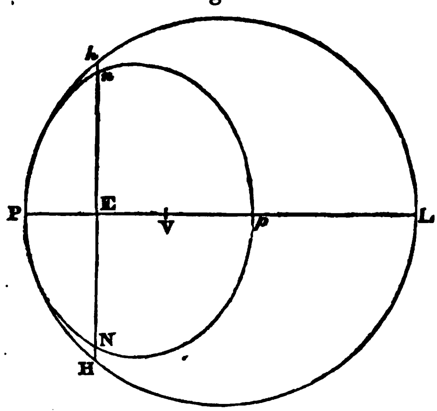
Now, when Pp is the transverse axis of an ellipse (fig. 2), since Pp is greater than LP and $Pp : PL = PE : PV$, therefore PE is greater than PV; and hence LV is always greater than LE; therefore NE^2 is greater than HE^2, also nE^2 is greater than hE^2; hence the circle falls wholly within the ellipse.

Fig. 3.



Again, when Pp is the transverse axis of an hyperbola (fig. 3), pE is greater than pP, and therefore LV is greater than LP, and consequently greater also than LE; hence NE^2 is greater than HE^2, and nE^2 greater than hE^2, and the circle is wholly within the hyperbola.

Fig. 4.



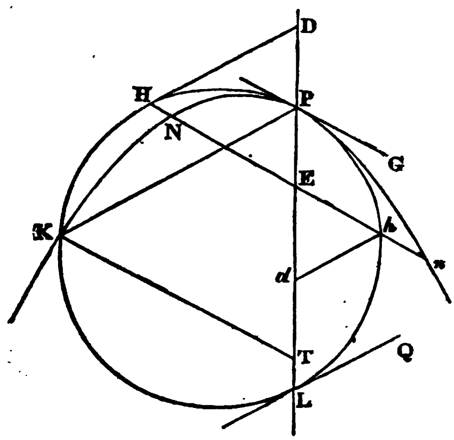
Lastly, when Pp is the conjugate axis of an ellipse (fig. 4), since pP is in this case less than LP, and pP : LP = PE : PV, therefore PE is less than PV; hence LV is less

than LE, and consequently NE^2 is less than HE^2, and nE^2 less than hE^2; therefore the circle is wholly without the ellipse.

PROPOSITION III.

The circle of curvature at the vertex of any diameter of a conic section which is not an axis, meets the conic section again in one point only; and between that point and the vertex of the diameter the circle falls wholly within the conic section on the one side, and wholly without it on the other.

Fig. 1.



Case 1. Let the conic section be a parabola, of which PL is a diameter, and let PLK be the circle of curvature at its vertex, cutting off from the diameter a segment PL equal to its parameter; draw PG touching the circle and parabola at P, and LQ touching the circle in L; also draw

PK parallel to LQ, meeting the circle in K, and KT parallel to PG, meeting the diameter in T. The lines KP, KT will be equal, and

$$KT^2 = PT \cdot LP \text{ (Lemma)};$$

therefore KT is a semi-ordinate to the diameter PL (Prop. 12, Part I.), and K is a point in the parabola. And since only one line PK can be drawn through P parallel to the tangent LQ, only one such point K can be found; therefore the circle of curvature cuts the parabola in one point besides the vertex of the diameter, and in no more. Between P and T draw NEn, any ordinate to the diameter PL, meeting the circle in H and h, and draw HD, hd parallel to the tangent LQ, meeting the axis in D and d; and because

$$NE^2 \text{ also } nE^2 = PE \cdot PL \text{ (Prop. 12, Part I.)}$$

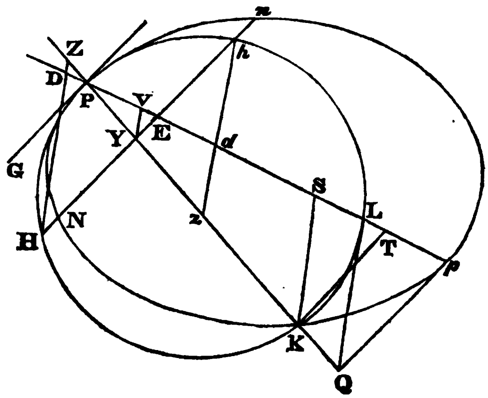
$$\left.\begin{aligned}\text{and } HE^2 &= PE \cdot LD \\ \text{and } h\,E^2 &= PE \cdot Ld\end{aligned}\right\} \text{(Lemma)},$$

$$\text{therefore } NE^2 : HE^2 = PL : LD,$$

$$\text{and } nE^2 : hE^2 = PL : Ld.$$

Now PL is less than LD; therefore NE is less than HE, and the circular arc PHK is without the parabola from the vertex to the intersection K. If the ordinate be more remote from the vertex than the position KT, then D and d will be both on the same side of the vertex, and therefore PL will be greater than LD, also greater than Ld, and consequently NE will be greater than HE, also nE than hE; hence it follows that the arc PLK falls wholly within the parabola.

Fig. 2.



CASE 2. Let the conic section be either an ellipse or an hyperbola, of which Pp is a diameter, and PLK the circle of equal curvature at its vertex, cutting off a segment PL equal to its parameter. Draw LQ touching the circle, and pQ touching the curve; and because this line is parallel to the line PG, which touches the circle and ellipse at P, the lines LQ, pQ make equal acute or obtuse angles (but in opposite directions) with pL (32, 3, and 29, 1, E.); therefore they will meet at a point Q. Join PQ; and because Q is without the circle, and P is in the circumference, the line PQ, which cannot be a tangent, must cut the circle in one other point K, and in no more. Draw KS and KT parallel to QL and Qp, meeting Pp in S and T.

Because of the parallels,

$$Pp : pT = PQ : QK = PL : LS \text{ (2, 6, E.)};$$

therefore (by alt.) $Pp : PL = pT : LS = pT \cdot TP : LS \cdot TP.$

$$\text{But } LS \cdot PT = KT^2 \text{ (Lemma)},$$

$$\text{therefore } Pp : PL = pT \cdot TP : KT^2.$$

Hence KT is a semi-ordinate to the diameter Pp, and K is a point in the ellipse or hyperbola (15 of Part II. and 23 of Part III.).

Draw NEn, any ordinate to the diameter, so as to meet the equicurve circle in H and h, and the line PQ in Y. Draw HDZ, hdz parallel to LQ, meeting Pp, PQ in D, Z and in d, z; also draw YV parallel to LQ, meeting Pp in V.

Because of the parallel lines,

$$\mathrm{P}p : \mathrm{E}p = \mathrm{PQ} : \mathrm{YQ} = \mathrm{PL} : \mathrm{VL};$$

$$\text{hence } \mathrm{P}p : \mathrm{PL}\ (= \mathrm{E}p : \mathrm{VL}) = \mathrm{PE}\cdot\mathrm{E}p : \mathrm{PE}\cdot\mathrm{VL}.$$

$$\text{But } \mathrm{P}p : \mathrm{PL} = \mathrm{PE}\cdot\mathrm{E}p : \mathrm{NE}^2 \text{ or } n\mathrm{E}^2,$$

$$\text{therefore } \mathrm{NE}^2 = n\mathrm{E}^2 = \mathrm{PE}\cdot\mathrm{VL}.$$

$$\left.\begin{array}{r}\text{But } \mathrm{HE}^2 = \mathrm{PE}\cdot\mathrm{LD}\\ \text{and } h\mathrm{E}^2 = \mathrm{PE}\cdot\mathrm{L}d\end{array}\right\}\ \text{(Lemma)},$$

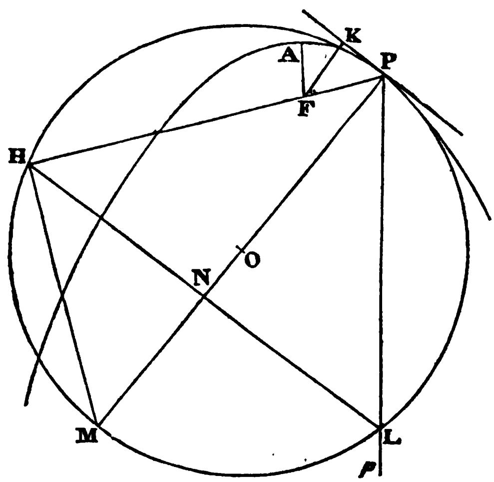
$$\text{therefore } \mathrm{NE}^2 : \mathrm{HE}^2 = \mathrm{VL} : \mathrm{LD} = \mathrm{YQ} : \mathrm{QZ},$$

$$\text{and } n\mathrm{E}^2 : h\mathrm{E}^2 = \mathrm{VL} : \mathrm{L}d = \mathrm{YQ} : \mathrm{Q}z.$$

Now wherever the point H be taken in the arc PHK, it is manifest that YQ will be less than QZ, therefore also NE is less than HE; thus the arc PHK falls wholly without the conic section. Again, since YQ always exceeds Qz, therefore nE always exceeds hE; hence the arc PhK falls wholly within the section.

PROPOSITION IV.

The chord of the circle of curvature which is drawn from the point of contact through the focus of a parabola is equal to that which is cut off from the diameter; and half the radius of the circle is a third proportional to the perpendicular from the focus upon the tangent, and the distance of the point of contact from the focus.



Let PL be the chord cut off from the diameter, and PFH the chord passing through F the focus; draw PM the diameter of the circle; join HL, HM, and draw FK perpendicular to the tangent at P. Because the lines PFH, PL make equal angles with the tangent at P (Cor. 3, 3, Part I.), the angles PHL, PLH are equal (32, 3, E.); hence PH = PL. Secondly, the triangles FKP, PHM being manifestly similar,

FK : FP = PH : PM = (Def. 9 and 1, Part I.) 4 PF : PM,

hence FK : FP = FP : $\frac{1}{4}$ PM, or $\frac{1}{2}$ the radius.

Cor. 1. Hence the radius is equal to $\frac{2FP^2}{FK}$.

Cor. 2. The radius is also equal to $\frac{2FK^3}{AF^2}$, where AF is the distance of the focus from the vertex of the parabola; for $FP = \frac{FK^2}{AF}$ (14, Part I.).

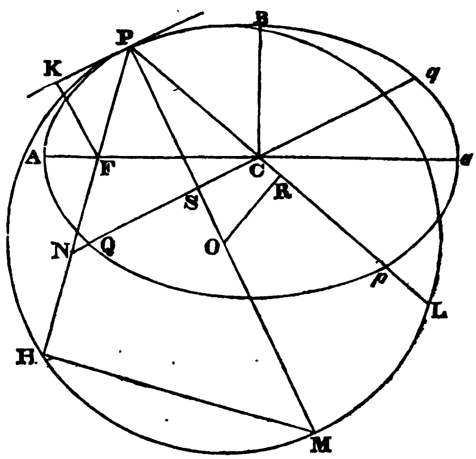
Cor. 3. Hence also the radius is equal to $\frac{\frac{1}{4}L \cdot FP^3}{FK^3}$, where

L denotes the parameter of the axis; for

$$\frac{2FP^2}{FK} = \frac{2AF \cdot FP^3}{AF \cdot FP \cdot FK} = \frac{\frac{1}{2}L \cdot FP^3}{FK^3}.$$

PROPOSITION V.

The radius of the circle of curvature at the vertex of any diameter of an ellipse or hyperbola is a third proportional to the perpendicular drawn from the centre upon the tangent and half the conjugate diameter; and the chord which is drawn from the point of contact through the focus is a third proportional to the transverse axis and conjugate diameter.



Let PL be the chord cut off from the diameter, and PFH the chord passing through F the focus; draw PM the diameter of the circle, and from the centre O draw OR perpendicular to PL, which will bisect PL in R; join HM, and draw the conjugate diameter QCq meeting PH in N and PM in S, then PS is equal to the perpendi-

cular from the centre C upon the tangent. The triangles PSC, PRO are similar; therefore

$$PS : PC = PR : PO;$$

but $PC : CQ = CQ : PR$ (Def. of param.),

therefore $PS : CQ = CQ : PO$.

Secondly, the triangles PSN, PHM are similar,

therefore $PN : PS = PM : PH$.

But $PS : CQ = (CQ : PO =)\ Qq : PM$,

therefore $PN : CQ = Qq : PH$;

or, since $PN = AC$ (Cor. to 19, Part II. and to 27, Part III.) $\quad Aa : Qq = Qq : PH$.

Cor. 1. Hence the radius of curvature is equal to $\frac{CQ^2}{PS}$, and the chord passing through the focus is equal to $\frac{2CQ^2}{AC}$.

Cor. 2. The radius of curvature is also equal to $\frac{CQ^3}{AC \cdot BC}$; for $PS = \frac{AC \cdot BC}{QC}$ (17, Part II.; and 25, Part III.)

Cor. 3. Draw FK from the focus perpendicular to the tangent, and let L denote the parameter of the transverse axis; the radius of curvature is also equal to $\frac{\frac{1}{2} L \times FP^3}{FK^3}$. For the triangles PFK, NPS are manifestly similar; therefore $FK : FP = PS : PN$, or $AC = BC : CQ$ (Prop. 17, Part II.);

$$\text{hence } CQ = \frac{FP}{FK} \times BC, \text{ and}$$

$$\frac{CQ^3}{AC \cdot BC} = \frac{FP^3}{FK^3} \times \frac{BC^2}{AC} = \frac{FP^3}{FK^3} \times \tfrac{1}{2} L.$$

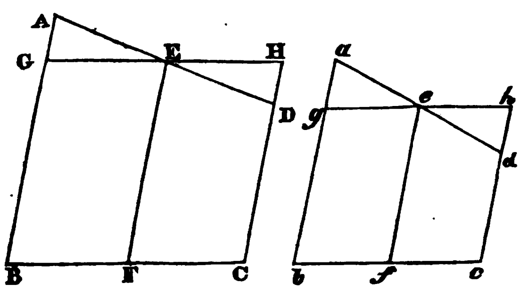
This expression for the radius of curvature is the same for all the three conic sections.

SECTION III.

AREAS OF THE CONIC SECTIONS.

LEMMA.

Let ABCD, *abcd*, two trapeziums, have each two parallel sides, and let the angles which the parallel sides AB, DC of the one figure make with its side BC be equal to the angles which *ab*, *dc*, the parallel sides of the other figure, make with its side *bc*; also, in the one figure, let EF, which is parallel to AB and DC, bisect the opposite sides BC, AD in F and E, and in the other figure let *ef*, a parallel to *ab* and *dc*, bisect *bc* and *ad* in *f* and *e*; the trapezium ABCD is to the trapezium *abcd* as the rectangle BC·EF to the rectangle *bc*·*ef*.

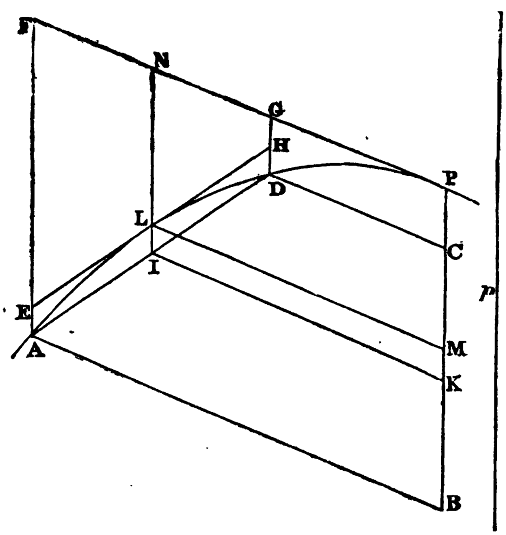


Through E and *e* draw GH and *gh* parallel to BC and *bc*, forming the parallelograms GC, *gc*. The triangles AEG, DEH are manifestly equal (26, 1, E.); therefore the trapezium ABCD is equal to the parallelogram GBCH. In the same way it appears that the trapezium *abcd* is equal to the parallelogram *gbch*. Now the parallelogram

GC has to the parallelogram *gc* the ratio compounded of the ratios of BG to *bg*, and of BC to *bc* (23, 6, E.); and the rectangle BC·BG has to the rectangle *bc·bg* the ratio which is compounded of the same ratios; therefore trap. ABCD : trap. *abcd* = BC·BG : *bc·bg* = BC·EF : *bc·ef*.

PROPOSITION I.

In a parabola, let ABCD *be a trapezium formed by* PB, *any diameter,* AB, CD *semi-ordinates to that diameter, and* AD *a chord in the curve; and let* EFGH *be another trapezium formed by* PF, *a tangent at the vertex of the diameter* PB, *by* AF, DG, *diameters produced at* A, D, *and* EH *a tangent parallel to the chord: the trapezium* ABCD *is double the trapezium* EFGH.



Let L be the point of contact of the tangent EH; draw a diameter through L, meeting the chord AD in I, and the tangent PF in N; draw LM a semi-ordinate to the diameter PB, and IK parallel to LM. Let p be the parameter of the diameter PB. And because AD is bisected in I, and BK : KC = AI : ID = FN : NG, therefore BC is bisected in K and FG in N.

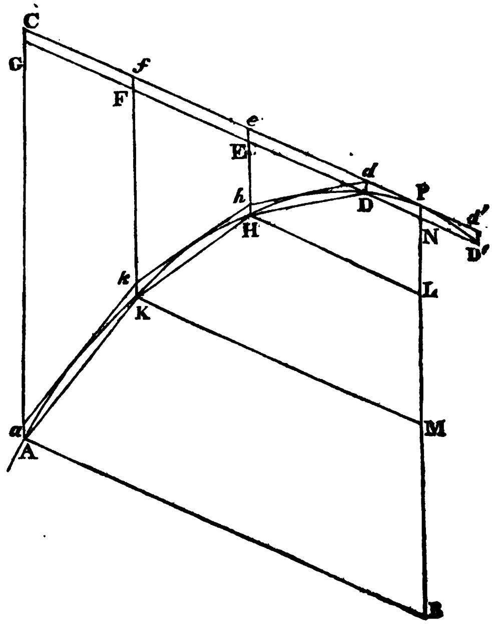
$$
\begin{aligned}
\text{And because } p \cdot PB &= AB^2 = PF^2, \\
\text{and } p \cdot PC &= DC^2 = PG^2; \\
\text{therefore } p \cdot BC &= PF^2 - PG^2 \\
&= (PF + PG)(PF - PG), \\
\text{that is, } p \cdot BC &= 2PN \cdot FG = 2LM \cdot FG. \\
\text{Now } p \cdot PM &= p \cdot LN = LM^2, \\
\text{therefore } p \cdot BC : p \cdot LN &= 2LM \cdot FG : LM^2, \\
\text{and } BC : LN &= 2FG : LM \text{ or } IK, \\
\text{and hence } BC \cdot IK &= 2FG \cdot LN.
\end{aligned}
$$

Now, by the premised lemma, the trapezium ABCD is to the trapezium EFGH as the rectangle BC · IK to the rectangle FG · LN; therefore the trapezium ABCD is double the trapezium EFGH.

Scholium. Since GH may be of any magnitude, the proposition will still be true when the points H and G coincide in the line PF.

PROPOSITION II.

Let AB *be a semi-ordinate to* PB, *any diameter of a parabola; complete the parallelogram* ABPC, *by drawing* PC *a tangent at the vertex, and* AC *parallel to* PB: *the space comprehended by* PA, *the arc of the parabola, and* PB, AB, *the absciss and ordinate, is two thirds of the parallelogram* PBAC.

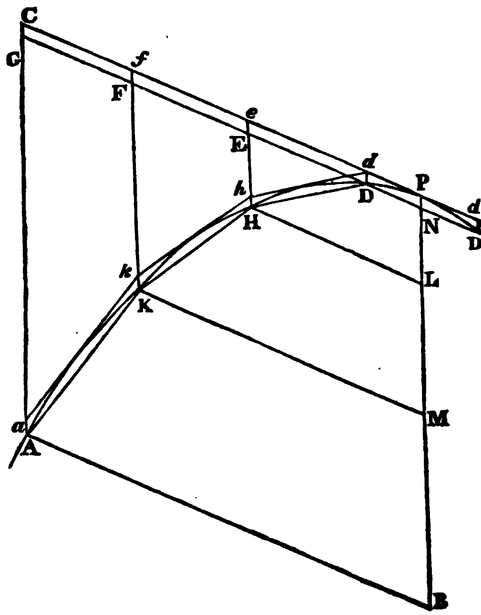


Divide PC into any odd number of equal parts, seven, for example; let P*d* be one, P*e* three, and P*f* five of these; take P*d'* equal to P*d*, and draw *d'*D', *d*D, *e*H, *f* K parallel to PB, meeting the curve in D', D, H, K; these lines, when produced, will be diameters; and they will

be equidistant, because they divide the line Cd' into equal parts. Because $Pd = Pd'$, therefore $dD = dD'$ (11, Part I.). Join DD', which will be parallel to dd' (33, 1, E.); let it meet PB in N; produce it to meet the other diameters in E, F, G. Draw HL, KM parallel to PC, also the chords DH, HK, KA, and, parallel to them, the tangents dh, hk, ka. And because the diameters whose vertices are D', D, H are equidistant, the tangents dd', hd will intersect each other at d, a point in the diameter passing through D (2 Cor. 15, Part I.); for a like reason, the tangents hd, hk will intersect in He, and hk, ak in FK.

The triangles HED, hed are in all respects equal, for $DH = dh$ (34, 1, E.) $DE = de$; and since $Hh = Dd = Ee$, therefore $HE = he$. The trapeziums KHEF, $khef$ are also equal; for HE, a side of the one, is equal to he, a side of the other, and HK, EF are equal to hk, ef respectively, and make equal angles with the equal sides HE, he between them; therefore, if the trapeziums be applied, one on the other, so that the equal sides HE, he coincide, the sides HK, hk will coincide, also EF, ef, and the trapeziums will entirely coincide. In like manner, the trapeziums AKFG, akfC are proved to be equal; and because the trapezium LHDN is double the triangle hde or its equal HDE (preceding Prop.), and the trapezium MKHL double the trapezium $fkhe$, or its equal FKHE, and the trapezium BAKM double $aCfk$, that is, double AGFK, the polygon NDHKAB will be double the polygon DHKAG; but these together make up the parallelogram ABNG, therefore the polygon NDHKAB, inscribed in the parabola, is two thirds of the parallelogram ABNG. Now the space bounded by the arc PA, the absciss PB, and semiordinate AB, exceeds the inscribed polygon; therefore it

also is greater than two thirds of the parallelogram ABNG or two thirds of the parallelogram ABPC, diminished by two thirds of the parallelogram PCGN.



The parallelograms HD*dh*, ED*de* are equal (35, 1, E.), so also are KH*hk*, E*ef*F, and AK*ka*, GF*f*C (36, 1, E.); therefore the sum of the four parallelograms A*k*, K*h*, H*d*, DP is equal to the parallelogram GNPC. Now if the polygon inscribed in the parabola be increased by these four parallelograms, there will be formed the polygon P*dhka*B, which exceeds the parabolic area; therefore that area is less than two thirds of the parallelogram ABNG increased by the whole parallelogram GNPC, and consequently less than two thirds of the parallelogram ABPC increased by one third of the parallelogram GNPC.

Let the parabolic space PAB be denoted by S, the parallelogram PCAB by R, and the parallelogram PNGC by V.

It has been proved that S $>$ $\frac{2}{3}$ R — $\frac{2}{3}$ V,

and that S $<$ $\frac{2}{3}$ R + $\frac{1}{3}$ V.

This is true whatever may be the magnitude of V; but the line Pd may be taken such that V may be less than any assignable space; therefore S can be equal to no assignable space that is either greater or less than two thirds of the space R, and consequently is exactly equal to two thirds of the space R.

DEFINITION.

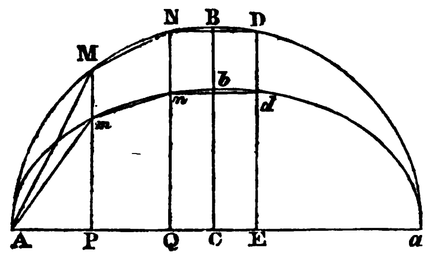
If the axes of an hyperbola be equal, it is called an *Equilateral* hyperbola.

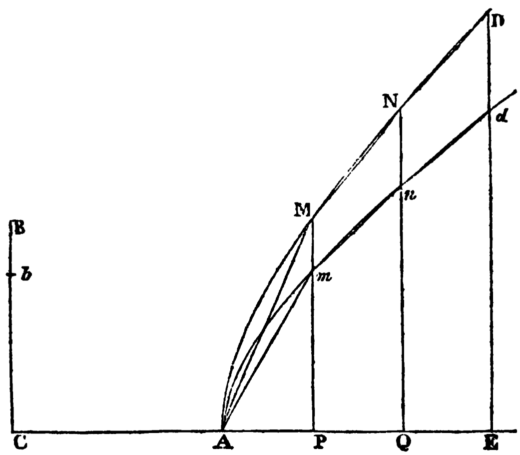
PROPOSITION III.

If two ellipses, or two hyperbolas, have a common transverse axis, and if from the same point in the axis there be drawn a semi-ordinate to each; the areas contained by the common absciss, the ordinates, and the curves, will be to each other as their conjugate axes.

Let AMBa Amba be two ellipses, and AMD, Amd two hyperbolas, which have each pair a common transverse axis Aa; and let BC, bc be their conjugate axes, and DE, dE semi-ordinates at the same point E in the common axis; the area AMDE is to the area AmdE as the axis BC to the axis bC.

Let the common absciss in both curves be divided into any number of equal parts AP, PQ, QE; draw semi-ordinates PmM, QnN, and the chords AM, MN, ND, Am, mn, nd.





In the case of two ellipses,

$$\left.\begin{aligned} MP^2 : AP \cdot Pa &= BC^2 : AC^2; \\ \text{and } AP \cdot Pa : mP^2 &= AC^2 : bC^2; \end{aligned}\right\} (13,\ 2)$$

$$\text{therefore, } \textit{ex. æq. } MP^2 : mp^2 = BC^2 : bC^2,$$

$$\text{and } MP : mP = BC : bC.$$

In the same way, in the two hyperbolas it may be proved that

$$MP : mP = NQ : nQ = DE : dE = BC : bC.$$

Now, in both curves (by 1, 6, E.),

$$\text{triangle } APM : \text{triangle } APm = PM : Pm = BC : bC;$$

$$\text{and since } MP : NQ = mP : nQ,$$

therefore, by composition and alternation,

$$MP + NQ : mP + nQ = NQ : nQ = BC : bC.$$

Now the area of a trapezium is known to be equal to the rectangle contained by the sum of the parallel sides and half the distance between them; therefore

$$MP + NQ : mP + nQ = \text{trap. } MPQN : \text{trap. } mPQn;$$

$$\text{and trap. } MPQN : \text{trap. } mPQn = BC : bC.$$

In the same way, it appears that the trapezium NQED is to the trapezium $nQEd$ as BC to bC; therefore (12, 5, E.) polygon AMNDE : polygon $AmndE$ = BC : bC.

This must be true, however great may be the number of sides of the polygon AMNDE, $AmndE$ inscribed in the curvilinear spaces; but, by a known principle in geometry, the limits of the polygons (which are the curvilinear spaces) must have the same ratio as the polygons themselves; therefore the curvilinear spaces AND, And have the same ratio as the semi-conjugate axes BC, bC.

Cor. 1. Hence it appears that the area of an ellipse is to that of its circumscribing circle as the conjugate axis to the transverse axis.

Cor. 2. It also appears that the area of any segment of an ellipse may be found from that of a corresponding segment of a circle; and the area of a segment of any hyperbola from the corresponding segment of an equilateral hyperbola.

APPENDIX.

SECTION I.

PROPERTIES OF THE PARABOLA AND ELLIPSE.

PROPOSITION I.

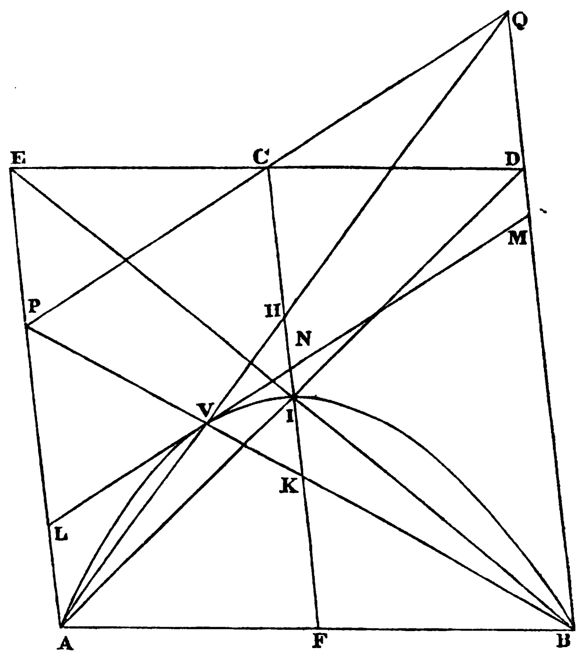
Let ABDE *be a parallelogram, given in position; through* C *the middle of one of its sides* ED, *draw any straight line* PCQ, *meeting the opposite sides in* P *and* Q; *and draw straight lines* PB, QA, *across the parallelogram to the ends of the remaining side, so as to intersect each other in* V; *the point* V, *and all points determined in the same way, are in a parabola.*

Draw AD, BE, the diameters of the parallelogram, intersecting in I; these will mutually bisect each other; I will therefore be a given point. Draw a straight line through I parallel to AE or BD; let it meet the lines AV, BV, and AB, in the points H, K, and F.

Because EC is equal to CD, and the angles PEC, PCE are respectively equal to the angles QDC, DCQ, the triangles CEP, CDQ are in every respect equal, and PE is equal to QD.

Because KI is parallel to PE, the triangles BIK, BEP are similar: Now BE is double BI, therefore EP is double IK. In

the same way it may be proved that QD is double IH; therefore IK is equal to IH.



The triangles BEA, BIF are similar, because IF is parallel to EA: Now BE is double BI, therefore BA is double BF, and AF is equal to FB. But it was shown that KI is equal to IH, therefore AF : FB = KI : IH. It has now been shown that the lines AV, BV, inflected to V from the ends of the given line AB, intercept in IF, a line given in position, segments IH, IK adjacent to a given point I, which have to each other the ratio of the segments into which that line divides AB, therefore (16, Part I.) the point V is in a parabola.

Cor. A straight line drawn through V, parallel to PQ, will touch the parabola at V. Let the line drawn through V meet the opposite sides of the parallelogram in L and M, and CF in N.

In the triangle BPQ, VM is drawn parallel to PQ, and, in the triangle BVQ, KH, which is parallel to BQ, meets the diverging lines VB, VM, VQ; therefore

$$BV : VP = BM : MQ = KN : NH \text{ (2, 6, E.)}$$

Hence $BV+VP : BV-VP = KN+NH : KN-NH$ (E. 5, E.) Now BE being bisected in I, and $BK : KP = BI : IE$, therefore BP is bisected in K, and $BV - VP = 2KV$: also because HK is bisected in I, $KN - NH = 2IN$: the proportion now becomes,

$$2BK : 2KV = 2KI : 2IN$$

$$\text{and } BK : KV = KI : IN;$$

hence VN is a tangent to the parabola. (Cor. 16, Part I.)

SCHOLIUM. By this proposition, having given AB any chord in a parabola, and FI the segment it cuts off from the diameter that bisects it; any number of points may be readily found in the curve; also, the lines which touch it at these points.

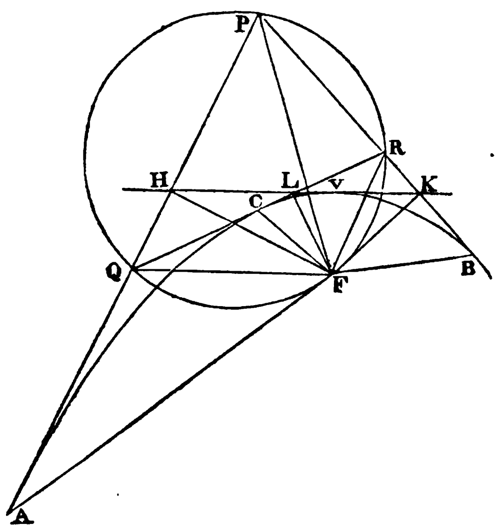
PROPOSITION II.

A circle described about a triangle, the sides of which touch a parabola, passes through its focus.[1]

Let PQR be a triangle, the sides of which touch a parabola at the points A, B, C, viz. PQ at A, PR at B, and QR at C, and let F be the focus; a circle described about the triangle PQR will pass through F.

Let a tangent at V, the vertex of the parabola, meet PQ in H, PR in K, and QR in L. Join FQ, FR, FL, FH, FK. The angles FHQ, FLQ are right angles (Cor. 1, 14, Part I.); therefore the points F, Q, H, L, are in the circumference of a circle, and the figure FQHL is a quadrilateral inscribed in that circle; hence the angle PQF or HQF is equal to the angle FLK (22, 3, E.)

[1] It may be proper to mention, that this proposition was given by the author of this work in Leybourne's *Mathematical Repository*, about the year 1797, because it has since that time appeared as new in the *Annales des Mathématiques*.



And because the angles FLR, FKR are right angles, the points FLRK are in the circumference of a circle; therefore the angles FLK, FRK in the same segment (21, 3, E.) are equal; but the angle PQF was shown to be equal to FLK; therefore PQF is equal to FRK; hence the quadrilateral PQFR is in a circle (22, 3, E.), and a circle described about the triangle PQR will pass through F.

Cor. 1. The angle which a straight line drawn from the focus of a parabola to the intersection of two tangents makes with either of them, is equal to the angle which a straight line drawn from the focus to the point of contact of the other tangent, makes with that tangent.

Let FP be drawn from F, the focus of a parabola, to P, the intersection of two tangents, and FA, FB to A, B, the points of contact: the angle FPB is equal to FAP, and FPA to FBP.

For, let a third tangent QR be drawn, meeting the other two in Q and R, and let a circle be described about the triangle PQR. Suppose now the point Q to approach to A, the line QR

will at last coincide with AP, and the angle FQR will become the angle FAP. But in every position of QR, the angle FQR is equal to FPB (21, 3, E.); therefore the angle FPB is equal to the angle FAP.

In the same way, it appears, that supposing R to approach to B, and at last to coincide with it, the angle FRQ, which (21, 3, E.) is always equal to FPA, becomes FBP; therefore the angles FPA, FBP are equal.

Cor. 2. The angles FPA, FBP being equal (Cor. 1), and the angles AFP, BFP also equal (5, Part I.), the triangles AFP, BFP are equiangular.

Cor. 3. If the focus of a parabola, and two tangents to the curve, be given in position, the points of contact are given.

For then the angles of each of the triangles AFP, BFP are given, and also their common side, viz. FP drawn from the focus to the intersection of the tangents; therefore all the sides of the triangles are given, and consequently the points A, B.

Cor. 4. If PA, PB be tangents to a parabola at A and B, and straight lines be drawn from F the focus, to P their intersection and A, B the points of contact; and QR be any third tangent, and straight lines be drawn from F to Q and R, the points in which it cuts the other two, the triangle FQR shall be similar to the triangles FAP, FBP.

For the angle FQR is equal to FPR, and FRQ to FPQ (21, 3, E.) which is equal to PBF (Cor. 1.); therefore the triangle QFR (4, 6, E.) is equiangular and similar to PFB or FAP (Cor. 2.)

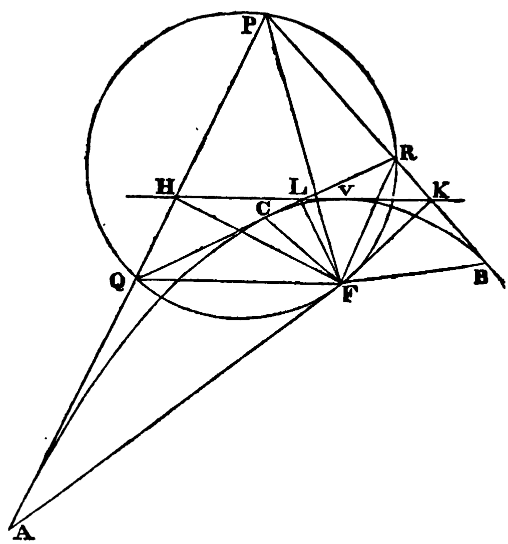
Cor. 5. If four straight lines given in position touch a parabola, the focus shall be given in position.

For, since each line must intersect all the others, they will form four triangles given in position, and circles described about these triangles will be given in position.

Now the focus is in the circumference of each circle, therefore they must all pass through the focus which will be given in position.

PROPOSITION III.

If three straight lines which intersect each other touch a parabola, the rectangle contained by the segments which each cuts off from the other two adjacent to the point of their intersection, is equal to the rectangle contained by the segments which it cuts off adjacent to their points of contact.



Let PQ, PR, QR be straight lines which touch a parabola at A, B, C: the rectangle QP · PR contained by the segments which QR cuts off from PA, PB adjoining to P, is equal to the rectangle QA · RB contained by the segments which it cuts off adjacent to the points of contact A, B.

Join FA, FB, FQ, FR. The triangles FAQ, FPR are equiangular (22, 3, E. and Cor. 1, 2), so also are the triangles FPQ, FBR; therefore

$$QA : RP \, (= QF : RF) = QP : RB,$$
$$\text{and } QP \cdot RP = QA \cdot RB.$$

In the same way it may be proved that these rectangles are equal, viz.

$$PQ \cdot QR = PA \cdot RC, \text{ and } PR \cdot RQ = PB \cdot QC.$$

Cor. 1. If two straight lines given in position touch a parabola at given points, any third tangent will cut off from them segments adjacent to given points which shall have a given ratio. Let PA, PB be the tangents given in position which intersect at P, and touch a parabola at given points A and B, and let QR, any third tangent, meet them in Q and R.

Because the rectangles $QP \cdot PR$ and $QA \cdot RB$ are equal,

$$QP : QA = RB : PR;$$

by composition $AP : QA = PB : PR$,

by alternation $AP : PB = QA : PR$.

Now, by hypothesis, A and P are given points, and AP, PB lines given in magnitude, therefore the lines QA, PR have to each other a given ratio, and they are cut off from AP, PB adjacent to given points A, P.

Cor. 2. If the points A and B are both on the same side of QR, then

$$PB \cdot PQ + PA \cdot PR = PA \cdot PB.$$

But if A and B are on contrary sides of QR, the difference of the rectangles $PB \cdot PQ$ and $PA \cdot PR$ is equal to the rectangle $PA \cdot PB$.

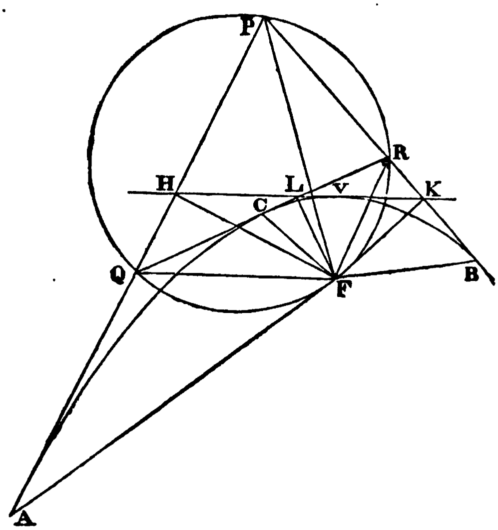
For, from the preceding corollary, $PA \cdot PR = PB \cdot QA$; and adding $PB \cdot PQ$ to these equals,

$$PB \cdot PQ + PA \cdot PR = PB \cdot QA + PB \cdot PQ = PB \cdot PA.$$

The other case is proved by subtracting the rectangle.

PROPOSITION IV.

Let PQA, PRB, QCR, *be three tangents to a parabola at the points* A, B, C, *and let* HK *be any fourth tangent which meets* PA *in* H, PB *in* K, *and* QR *in* L; *the lines* QR, HK *are similarly divided, the former at* C, *the point in which it touches the parabola, and the latter at* L, *the point in which it intersects* QR.



Let V be the point in which HK touches the parabola.

$$\text{Because } HL \cdot QL = HV \cdot QC \text{ (Prop. 3)},$$
$$\text{and } KL \cdot RL = KV \cdot RC;$$
$$\text{therefore } HV : HL = QL : QC,$$
$$\text{and } KL : KV = RC : RL.$$

From the first of these, by division, $LV : LH = CL : QC$;
From the second, by conversion, $KL : LV = RC : CL$;
Therefore, *ex æquali*, $KL : LH = RC : QC$.

Thus it appears that the lines QR, HK are similarly divided at C and L. In the same way it may be proved that

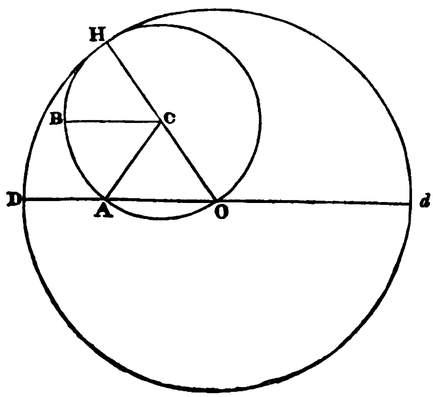
$$HL : HK = AQ : AP, \text{ and}$$
$$HK : LK = BP : BR.$$

Cor. If four or any greater number of straight lines given in position touch a parabola, any tangent to the curve which intersects them all will be divided by them into segments which have to each other given ratios.

For, if four or more tangents be given in position, the focus and the points of contact of all the tangents will be given; therefore the ratio of every three adjoining segments of the indeterminate tangent will be given.

PROPOSITION V.

If a moveable circle roll along the concave circumference of a fixed circle in the same plane, and the radius of the former be half that of the latter; any given point in its circumference will describe a diameter of the fixed circle.



Let ABHO, a moveable circle (which may be called the *generating circle*), whose centre is C, roll along DH*d* a fixed circle, of which O is the centre, both being in the same plane, and let the radius CH be half of the radius OH; any given point A in the circumference of the generating circle will always be in D*d*, some diameter given in position of the other circle.

Let H be the point of contact of the circles: the points H, C, O are in a straight line (11, 3, E.); and because the diameter of the inner circle is half that of the outer circle, one of its extremities will always be at O, the centre of the fixed circle.

Suppose that at the beginning of the motion the point A was at D, a given point in the circumference of the fixed circle, and that by rolling along the arch DH, the generating circle has come to the position OAH: draw AC to its centre, and bisect the angle ACH, and consequently the arch AH, by the radius BC.

The arch DH is equal to the arch AH, because every element of the one has been applied to an equal corresponding element of the other: therefore the arch DH is double the arch BH, and the radius of the circle DH is by hypothesis double the radius of the circle BH. Now, in different circles, equal angles at their centres stand on arches which have the same ratio as their circumferences or their radii, therefore an angle at O, the centre of the fixed circle, standing on the arch DH, will be equal to the angle BCH, that is, to the angle AOH (20, 3, E.); hence a straight line drawn through O and A will pass through the given point D; thus A will always be in the diameter DO*d*, which is given by position, and by the motion of the circle will describe that diameter.

Cor. 1. A diameter DO*d* drawn through the moving point A, in any one position, will be its *Locus* in every position.*

Cor. 2. The generating circle will have made two complete revolutions about its centre C when its diameter has completed one revolution about the centre O.

Cor. 3. When the generating circle has made a complete revolution about O, every point in its circumference will have described a diameter, passing twice through the centre, and have returned to its first position.

Scholium. The refined notion of generating lines by supposing a curve to roll along a straight line or curve, is due to the moderns.

Galileo appears to have been the first who introduced it into geometry, and in this way he indicated the *cycloid*, the discussion of which by Mersenne, Descartes, Pascal, and others, was

* A line, of which every point satisfies a prescribed condition of a geometrical hypothesis, is called the *Locus* of that condition. Thus, a Parabola is the locus of all points which are at the same distance from a given point and a straight line given in position: and an ellipse is the locus of the vertical angle of a triangle whose base is given in position and magnitude, and the sum of its sides equal to a given line.

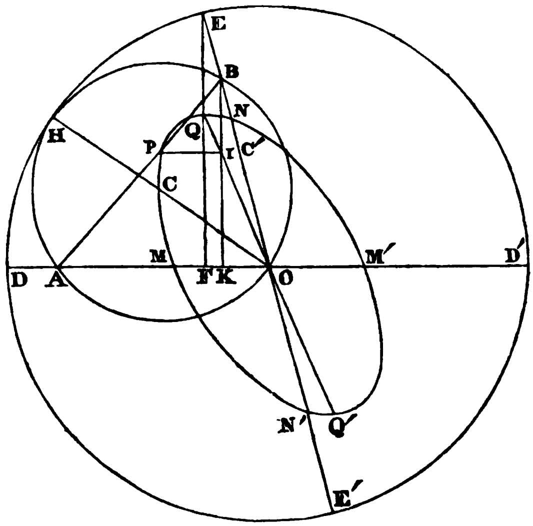
the beginning of that series of discoveries which has since gradually expanded into the modern geometry. The ancients, however, in some cases employed motion in the formation of geometrical figures. Euclid defined a sphere to be the solid figure described by the revolution of a semicircle about its diameter, which remains unmoved; and Archimedes defined his spiral by the uniform motion of a point along a straight line which at the same time turns with an uniform angular motion about one of its extremities. The preceding proposition, and the following, may be established by the ordinary method of geometrical reasoning; or instead of supposing one circle to roll on another, we might suppose the diameter of the generating circle to turn about the centre of the fixed circle with an uniform angular motion, while at the same time it turned uniformly about its own centre, so as to make two complete revolutions in the time its diameter makes one. It then might be easily shown, that the extremity of any radius of the revolving circle would describe a diameter of the fixed circle.

The property here demonstrated is elegant, and remarkable in having been applied in mechanics to the production of a reciprocating rectilinear motion by means of a rotatory motion.

PROPOSITION VI.

If a moveable or generating circle roll along the concave circumference of a fixed circle in the same plane, and the radius of the former be half that of the latter, as in the preceding proposition; any given point in the plane of the generating circle, within or without it, will describe an ellipse, of which conjugate diameters will be given in position.

Let DHED′E′ be the fixed circle, O its centre, and AHBO the generating circle, which rolls along the concave circumference DHE: any given point P (the generating point) in the



plane of this circle, within or without it, will describe an ellipse given in position.

Take a given point A anywhere in the circumference of the generating circle, and draw a straight line through A, and the generating point P, meeting the circumference in B: Thus A and B will be given points in the circle, and AP, PB lines given in magnitude. Draw OD, OE, radii of the fixed circle, through the points A and B: these will be lines given in position. (Preceding Prop.)

Let OCH be the revolving diameter of the generating circle, and C its centre. When OH has made a complete revolution about the centre O, the point A will have been twice at O, and BP, one of the segments of AB, will have coincided entirely with OM, OM′, equal segments of the diameter DD′, on opposite sides of the centre; therefore, if in DD′ there be taken OM and OM′, each equal to the line BP, M and M′ will be given points in which the locus of P cuts the line DD′.

For a like reason AP, the other segment of AB, will have co-

incided with segments of the diameter EE′ in two opposite positions, viz. ON, ON′; therefore, if ON and ON′ be taken each equal to AP, N and N′ will be also given points in which the locus of P intersects the line EE′.

By the motion of the generating circle, the point B, either of the extremities of the revolving chord AB, will in the course of a revolution have come to E, a point in the circumference of the fixed circle. The angle OAB will then be in a semicircle (of the generating circle), and AB will have the position of a perpendicular to OD. Let this be the line EF, which will be given in position, and also in magnitude, because it is equal to the chord BA. Take FQ = AP, therefore QE = PB; then Q will be a fifth given point in the locus of P. Join QO, and take OQ′ = OQ: the line QQ′, which is bisected at O, will be given in position and magnitude. Draw BK perpendicular to OD, and PI parallel to it, meeting BK in I. The triangles BPI, BAK are similar (4, 6, E.), therefore

$$\mathrm{BI} : \mathrm{IK} = \mathrm{BP} : \mathrm{PA} = \mathrm{EQ} : \mathrm{QF} :$$

Because the lines BK, EF are similarly divided at I and Q, the points O, I, Q are in a straight line (Lemma to Prop. 8, Part I.); so that I is in the line QOQ′.

$$\text{Again, because } \mathrm{EO} : \mathrm{OB} = \mathrm{QO} : \mathrm{OI},$$

and that OB cannot exceed OE, therefore OI cannot be greater than OQ; and when AB is not perpendicular to DD′, the point I will always be between Q and Q′: in no case can it be beyond these limits.

The triangles OQE, OIB, are similar (4, 6, E.), therefore their sides are proportionals, and

$$\mathrm{OQ}^2 : \mathrm{OI}^2 = \mathrm{QE}^2 \text{ or } \mathrm{BP}^2 : \mathrm{BI}^2,$$

and, by division, $\mathrm{OQ}^2 : \mathrm{QI} \cdot \mathrm{IQ}' = \mathrm{BP}^2 : \mathrm{PI}^2$;

hence, by alternation, and observing that BP = OM,

$$\mathrm{OQ}^2 : \mathrm{OM}^2 = \mathrm{QI} \cdot \mathrm{IQ}' : \mathrm{PI}^2 ;$$

therefore the point P is in an ellipse, of which QQ′ and MM′, lines given in magnitude and position, are conjugate diameters (converse of 13, Part II.)

Cor. 1. The ellipse described by the generating point P depends entirely on the magnitude of the generating circle AOB, and the position of the point in its plane, that is, on OC the radius of the circle, and CP, the distance of P from the centre; therefore, whatever be the position of the revolving chord AB, if it always pass through the same point P, the ellipse will be the same, and have the same position on the plane of the fixed circle.

Cor. 2. The position of DD′, EE′, the diameters of the fixed circle, which are the loci of the extremities of the revolving chord AB, and consequently the position of MM′, NN′, the diameters of the ellipse, depend entirely on the position of the chord in respect of the centre; they will be different for different chords, but for the same chord they will have a fixed position. This is evident from the last proposition.

Cor. 3. The semidiameters OM, ON of the ellipse in which (produced if necessary) the revolving chord terminates, are equal to the distances of the generating point P from the ends of the chord, viz. OM to PB, and ON to PA, and the angle which the diameters MM′, NN′ make at the centre is half the angle which the chord subtends at the centre of the generating circle; for in the course of a revolution of the generating circle about the centre O, the segments PB, PA of the chord will have been applied upon the lines OM, ON, so as entirely to coincide with them: the rest is evident (20, 3, E.).

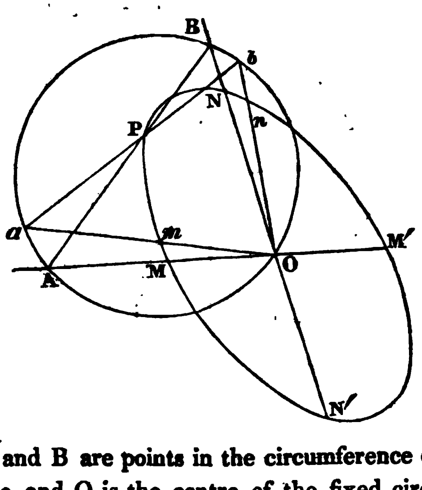
Cor. 4. When the revolving chord AB comes into the position EF, a perpendicular to MM′, either of the diameters, which are the loci of its extremities, it then passes through Q, a vertex of the diameter QOQ′, which is the conjugate of MM′, and in this position it is a normal to the ellipse; for a tangent to the ellipse at Q is parallel to MM′, and therefore perpendicular to EF.

Scholium. From this proposition it appears that if the ends of a straight line AB of a given length be carried along two straight lines DOD′, EOE′, given in position; any point P in AB

(or in AB produced), at given distances from its extremities, will describe an ellipse, the centre of which will be at the intersection of the lines given in position. It is upon this principle that elliptic compasses and lathes for turning ovals are constructed. An instrument for describing ellipses will be described farther on.

PROPOSITION VII.

Supposing the ellipse MPN, *whose centre is* O, *to have been described according to the hypothesis of Prop. VI.; let* AOB *be the generating circle in any position on the plane of the fixed circle, and* P *the generating point, which is also a point in the ellipse; through* P *draw any chord* APB; *join* OA *and* OB; *take* OM *and* OM′ *in contrary directions, each equal to* PB, *and* ON *and* ON′ *in contrary directions, each equal to* PA; *then* MM′, NN′ *will be two diameters of the ellipse.*



Since A and B are points in the circumference of the generating circle, and O is the centre of the fixed circle, the lines AO, BO will have the same position in the ellipse for all positions of the chord AB, because they are the loci of the points

A, B; therefore (6 Cor. 3) the semidiameters OM, ON will be respectively equal to PB and PA, the distances of P from the ends of the chord AB; hence if OM, ON be taken equal to PB and PA, the points M, N will be the vertices of diameters of the ellipse.

Cor. And if other chords *ab*, &c. be drawn through P, and O*a*, O*b*, &c. be joined, and there be taken O*m* equal to *b*P, and O*n* equal to P*a*, &c. then *m*, *n*, &c. will be points in the ellipse; and in this way any number of points whatever may be found from a single position of the generating circle.

PROPOSITION VIII.

Supposing an ellipse to be described according to the hypothesis of Prop. VI.; if the generating point be within the generating circle (fig. 1), half the sum of its semiaxes is equal to the radius of the circle; and half their difference, to the distance of the generating point from its centre. But if the point be without the circle (fig. 2), then half the difference of the semiaxes is equal to its radius, and half their sum to the distance of the generating point from its centre.

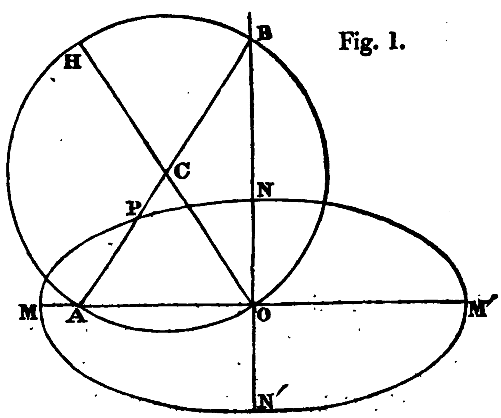
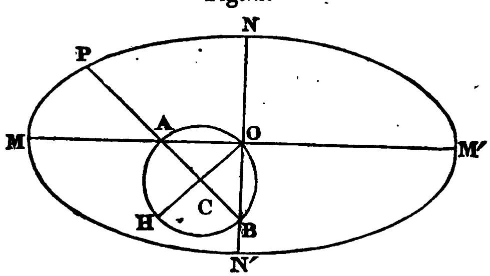


Fig. 1.

Let AOB be the generating circle (fig. 1 and 2), and P the generating point at any point in the curve. It appears from Prop. VI. Cor. 4, that a chord in the circle passing through P and A, the intersection of the circle, and MM′, one of the axes, will also p ss through B, the point in which it meets the other axis NN′; and in this case the chord will pass through C the centre (31, 3, E.), because the axes form right angles at the centre of the ellipse. Therefore, when the point P is within the circle AOB (fig. 1), the radius OC or AC is half the sum of BP and AP, that is, of OM and ON, the semiaxes (7); and CP, the distance of the generating point from the centre, is half the difference of PA and PB, or of OM and ON.

Fig. 2.



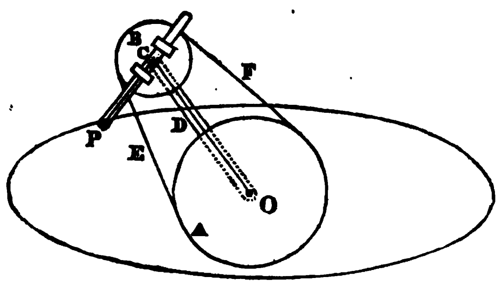
When the generating point P is without the circle AOB (fig. 2), then the radius OC or AC is manifestly half the difference of AP and BP; and CP, the distance of the generating point from the centre, is half the sum of AP and BP, that is, of OM and ON, the semiaxes.

Cor. 1. Hence it appears that the same ellipse may be described by two different generating circles, each rolling on its own fixed circle, viz. by one whose diameter is the sum of the semiaxes, and by another whose diameter is their difference; in the first way the generating point will be within the circle, and in the second without it.

Cor. 2. Also it appears that the chord of the generating circle, intercepted between the diameters of the ellipse that

pass through its extremities (and which passes through the generating point), is equal to the sum of the semidiameters when the generating point is within the circle, but to their difference when the point is without the circle.

Scholium. The curves which may be generated by a point in the plane of a moveable circle which rolls along the circumference of a fixed circle, are called cycloids, also epicycloids. They are of two kinds, one generated by a circle rolling on the convex circumference, and another by its rolling on the concave. Some writers confine the name *epicycloids* to the first class, and call the second *hypocycloids*. It appears from this proposition that an ellipse is an hypocycloid.



The property in question has suggested an instrument for generating an ellipse elegantly, by continued motion. A and B are two wheels, the axes of which turn in holes C, O, near the ends of the connecting bar D. One of the wheels B must be just half the diameter of the other A, which may be of any size, and a band EF goes round them outside; an arm CP is attached to the wheel B, and admits of being lengthened or shortened by sliding along its surface in a socket, which may be anywhere on the wheel. Suppose now that the wheel A is fixed or kept from turning, and that the bar D is turned round the centre O, carrying at its other extremity the wheel B; the action of the band EF will then turn this wheel B round its centre C, and while the bar makes one revolution round the centre

of the fixed wheel, the other wheel will make two revolutions about its centre.

The use of the sliding arm CP is to give extension to the surface of the wheel, so that P, any point in the arm, may be regarded as a point in the plane of a circle turning about a moveable centre C, while that centre revolves about a fixed centre O. From this description it is easy to see that C, the centre of the wheel B, may be regarded as the centre of a circle which rolls on the inside of a circle whose centre is O; also that any point P in the plane which is the extension of the surface of the wheel, is just a point in the plane of the rolling circle; and since the circle of which C is the centre makes two turns in going round that of which O is the centre, the radius of the one circle must be double that of the other; and hence it follows from the proposition that the path of the point P in space is an ellipse.*

* By the application of this principle, a *Planetarium* or *Cometarium* may be made, which shall exhibit to the eye the motion of a planet or comet in an elliptic orbit about the sun. Some instruments have actually been made which represented the motion of Encké's comet; the wheels were concealed by a cover, on which a ball representing the sun was placed in the focus of the ellipse.

SECTION II.

EQUATIONS OF THE CONIC SECTIONS.

The nature of a curve may be expressed by algebraic equations in various ways, particularly in two. Taking the circle as an example: If a perpendicular be drawn from any point in the circumference to the diameter, the square of that perpendicular is equal to the rectangle contained by the segments into which it divides the diameter; and this relation is the same for all points of the curve. Now, if the diameter of the circle be denoted by a, the perpendicular by y, and its distance from one end of the diameter by x; then its distance from the other end will be $a - x$, and the property in question may be expressed by an equation, viz. $y^2 = ax - x^2$, which is indeterminate, since a being constant, x and y may have any values whatever consistent with the conditions to be satisfied. This equation expresses the nature of a circle, and distinguishes it from all other curves; and is therefore called an *Equation of a Circle.* In the same way, each of the three conic sections has its peculiar equation, which distinguishes it from the others. This form is called an *Equation of rectangular Co-ordinates.*

Another way of expressing the nature of a curve is by assuming a fixed point in a line having a determinate position, and conceiving the curve to be generated by a straight line which turns about that point as a pole. If now the revolving line, called the *radius vector*, be denoted by the letter r, and the variable angle which it makes with the line given by position by v; then an algebraic equation involving r, v, and constant quantities, will express the nature of the curve, and is called a *Polar Equation.*

If, for example, the angle v and the line r are such that, a being a constant quantity, in every position $r = a \cos. v$; the extremity of the revolving line will describe a circle; and this is a

polar equation to a circle, of which a is the diameter, and the pole is at one of its extremities.

EQUATIONS OF RECTANGULAR CO-ORDINATES.

In the parabola, let us suppose that the origin of the rectangular co-ordinates is at the vertex of the axis; let p be its parameter, y any ordinate to the axis, and x the absciss; then, from the property of the curve demonstrated in 12, Part I.

$$px = y^2 \text{.....................(P).}$$

This is an equation of the parabola.

In the ellipse and hyperbola, let a be the semi-transverse axis, b the semi-conjugate, y any semi-ordinate to the transverse axis, and x the distance from the ordinate to either vertex; then,

In the ellipse, $a^2 : b^2 = x\,(2a - x) : y^2$ (13, Part II.)

therefore $a^2y^2 = 2ab^2x - b^2\,x^2$.................$(E_1)$.

In the hyperbola, $a^2 : b^2 = x\,(2a + x) : y^2$ (21, Part III.)

therefore $a^2y^2 = 2ab^2x + b^2x^2$..........(H_1).

These are equations of the curves.

In the ellipse and hyperbola, the centre may be taken as the origin of the co-ordinates; and then, putting x' the distance of the centre from the ordinate, we have $x = a - x'$ in the ellipse, and $x = x' - a$ in the hyperbola; therefore,

In the ellipse, $a^2y^2 = a^2b^2 - b^2x'^2$, and $\dfrac{x'^2}{a^2} + \dfrac{y^2}{b^2} = 1$...........$(E_2)$.

In the hyperbola, $a^2y^2 = b^2x'^2 - a^2b^2$, and $\dfrac{x'^2}{a^2} - \dfrac{y^2}{b^2} = 1$.....$(H_2)$.

These are also equations of the curves.

Instead of b, the semi-conjugate axis, we may introduce the parameter p into the equations; for, putting p for the parameter,

In the ellipse, $x\,(2a - x) : y^2 = 2a : p$...........(15, Part II.)

In the hyperbola, $x\,(2a + x) : y^2 = 2a : p$.....(23, Part III.)

Hence, in the ellipse, $y^2 = px\left(1 - \frac{x}{2a}\right)$...............($E_8$),

in the hyperbola, $y^2 = px\left(1 + \frac{x}{2a}\right)$.........($H_8$).

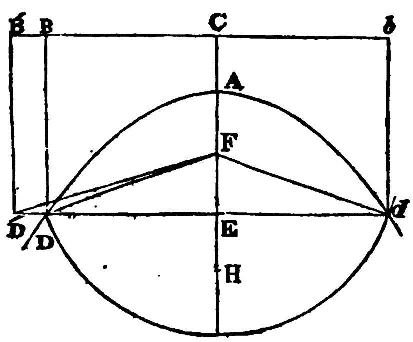
From what has been shown, it appears that the rectangular co-ordinates of a conic section are in every case the variable quantities of an equation of the second degree; and hence it follows that every conic section is a line of the *second order*.

We have supposed that the ordinates are drawn to the axis of a parabola, and to the transverse axis of an ellipse or hyperbola; but the equations hold true of ordinates to any diameter in each curve.

POLAR EQUATIONS.

The Parabola.

Let AH be the axis of a parabola, A the vertex, F the focus, and FD a line drawn to D any point in the curve. Draw DE perpendicular to the axis, and let BC be the directrix. Put p = parameter, r = FD the radius vector, and v = angle AFD.



By the definition of the curve, $r = EC = FC \mp EF = \frac{1}{2}p \mp EF$; the negative sign to be taken when the angle DFC is acute, but the positive when it is obtuse.

Now, by trigonometry, $EF = r \cos. v$; therefore, observing that the cosine is positive when the angle v is acute, but negative when it is obtuse, we have $r = \frac{1}{2}p - r \cos. v$; hence, and because $1 + \cos. v = 2 \cos.^2 \frac{1}{2} v$,

$$\cos. v = \frac{\frac{1}{2}p - r}{r} \ldots\ldots(P_1), \qquad r = \frac{\frac{1}{4}p}{1 + \cos. v} \ldots\ldots(P_2).$$

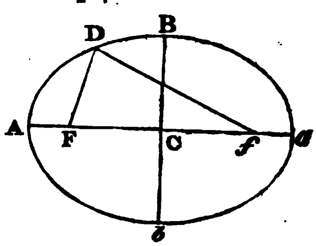
These are *polar equations* of the parabola.

The Ellipse and Hyperbola.

The polar equations of the ellipse and hyperbola may be elegantly deduced from the definitions of the curves, by two known theorems in plane trigonometry, as follows:

The sides of any plane triangle being p, q, r, and u the angle opposite to p, and v its supplement, it is proved in elementary works on Trigonometry (Playfair's *Geometry*, Prop. 7 and 8 of the *Trigonometry*) that if $s = \frac{1}{2}(p + q + r)$,

$$\sin.^2 \tfrac{1}{2} u = \frac{(s-q)(s-r)}{qr}, \qquad \cos.^2 \tfrac{1}{2} u = \frac{s(s-p)}{qr}$$



Let Aa be the transverse axis of an ellipse or hyperbola, F and f the foci, and C the centre: suppose the curves to be generated by the rotation of the radius vector $FD = r$ about F as a pole: join Df: let F be the focus nearest to A.

1. In the ellipse put u for the angle DFf, and v for the angle DFA; also put $D = Fa$, $d = FA$: then $Aa = D + d$, $Ff = D - d$, and because $Df + DF = Aa$, that is, $Df + r = D + d$, therefore $Df = D + d - r$. Let us now assume that $Df = p$, $Ff = q$, then

$$\left.\begin{array}{r} p = D + d - r, \\ q = D - d, \\ r = r, \\ \frac{1}{2}(p+q+r) = s = D; \end{array}\right\} \text{ and hence } \left\{\begin{array}{l} s - p = r - d, \\ s - q = d, \\ s - r = D - r. \end{array}\right.$$

These values being substituted in the trigonometrical formulæ, and observing that $\sin. \frac{1}{2} u = \cos. \frac{1}{2} v$, we have

$$\cos.^2 \tfrac{1}{2} v = \frac{d}{D-d} \cdot \frac{D-r}{r} \ldots\ldots\ldots\ldots\ldots\ldots (E_1),$$

$$\sin.^2 \tfrac{1}{2} v = \frac{D}{D-d} \cdot \frac{r-d}{r} \ldots\ldots\ldots\ldots\ldots\ldots (E_2);$$

and because $\tan.^2 \frac{1}{2} v = \dfrac{\sin.^2 \frac{1}{2} v}{\cos.^2 \frac{1}{2} v}$, therefore

$$\tan.^2 \tfrac{1}{2} v = \frac{D}{d} \cdot \frac{r-d}{D-r} \ldots\ldots\ldots\ldots\ldots\ldots (E_3);$$

also because $\cos. v = \cos.^2 \frac{1}{2} v - \sin.^2 \frac{1}{2} v$, therefore

$$\cos. v = \frac{2Dd - (D+d) r}{(D-d) r} \ldots\ldots\ldots\ldots\ldots\ldots (E_4).$$

Reversely we find from (E_1), (E_2), and (E_4),

$$\frac{D}{r} = 1 + \frac{D-d}{d} \cos.^2 \tfrac{1}{2} v \ldots\ldots\ldots\ldots\ldots\ldots (E_5),$$

$$\frac{d}{r} = 1 - \frac{D-d}{D} \sin.^2 \tfrac{1}{2} v \ldots\ldots\ldots\ldots\ldots\ldots (E_6),$$

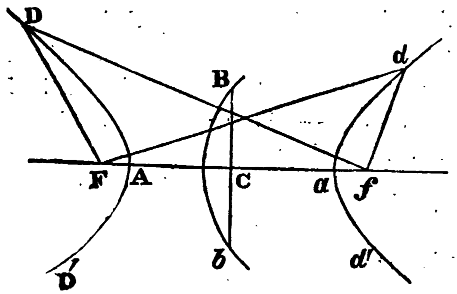
$$r = \frac{2Dd}{D + d + (D-d) \cos. v} \ldots\ldots\ldots\ldots\ldots\ldots (E_7).$$

If a denote the semi-transverse axis, and e the eccentricity, then, observing that $2a = D + d$, and $2e = D - d$, and $Dd = a^2 - e^2$,

$$r = \frac{a^2 - e^2}{a + e \cos. v} \ldots\ldots\ldots\ldots\ldots\ldots (E'_7).$$

These are the *polar equations* of the ellipse, referred to a focus. The formulæ are neat; and may be useful, although I have not observed that any besides (E_4) and (E_7) have hitherto been given in treatises on conic sections.

2. In the hyperbola, Df — DF = Aa, that is, Df — $r = D - d$; therefore D$f = D - d + r$. Assume now that D$f = p$, F$f = q$,



DF $= r$, and put $v =$ the angle DF*f*. From the nature of the curve,

$$\left.\begin{aligned} p &= r + D - d, \\ q &= D + d, \\ r &= r, \\ \tfrac{1}{2}(p+q+r) &= s = D + r; \end{aligned}\right\} \text{and hence} \left\{\begin{aligned} s - p &= d, \\ s - q &= r - d, \\ s - r &= D. \end{aligned}\right.$$

These values, substituted in the trigonometrical formulæ, give

$$\cos.^2 \tfrac{1}{2} v = \frac{d}{D+d} \cdot \frac{D+r}{r} \ldots\ldots\ldots\ldots (H_1),$$

$$\sin.^2 \tfrac{1}{2} v = \frac{D}{D+d} \cdot \frac{r-d}{r} \ldots\ldots\ldots\ldots (H_2),$$

$$\tan.^2 \tfrac{1}{2} v = \frac{D}{d} \cdot \frac{r-d}{D+r} \ldots\ldots\ldots\ldots (H_3),$$

$$\cos.\ v = \frac{2Dd - (D-d)r}{(D+d)r} \ldots\ldots\ldots\ldots (H_4).$$

Again, from these we obtain,

$$\frac{D}{r} = \frac{D+d}{d} \cdot \cos.^2 \tfrac{1}{2} v - 1 \ldots\ldots\ldots\ldots (H_5),$$

$$\frac{d}{r} = 1 - \frac{D+d}{D} \sin.^2 \tfrac{1}{2} v \ldots\ldots\ldots\ldots (H_6),$$

$$r = \frac{2Dd}{D - d + (D+d)\cos.\ v} \ldots\ldots\ldots\ldots (H_7);$$

and, putting a for the transverse axis, and e for the eccentricity, and observing that $Dd = e^2 - a^2$,

$$r = \frac{e^2 - a^2}{a + e\cos.\ v} \ldots\ldots\ldots\ldots (H_7).$$

These are the *polar equations* of the hyperbola referred to a focus; and it may be remarked, that if the algebraic sign of D in the like equations of the ellipse be changed, that is, if $+D$ be changed to $-D$, and $-D$ to $+D$, they will become identical with those of the hyperbola.

If $Aa = D + d$, the transverse axis of an ellipse, be exceedingly great in respect of the radius vector r, or its least value d, then $\frac{D-r}{D-d}$ will be almost $= 1$; and the ellipse will nearly coincide with a curve whose equations are

$$\cos.^2 \tfrac{1}{2} v = \frac{d}{r}, \quad \sin.^2 \tfrac{1}{2} v = \frac{r-d}{r}, \quad \tan.^2 \tfrac{1}{2} v = \frac{r-d}{d},$$

$$\cos. v = \frac{2d-r}{r}, \quad r = \frac{2d}{1+\cos. v} = \frac{d}{\sin.^2 \frac{1}{2} v}.$$

These are manifestly the polar equations of a parabola of which the parameter $= 4d$, the radius vector $= r$, and the angle it makes with a line joining the focus and vertex $= v$; hence it appears that if d, the least distance of the focus of an ellipse from the vertex, be supposed to remain constant, and the greatest, D, to increase continually, the form of the ellipse will approach to that of a parabola; and the same is true of an hyperbola, because one of the opposite hyperbolas will on that hypothesis recede continually from the other. Therefore, in the language of modern analysis, a *parabola may be regarded as an ellipse or hyperbola, of which the distance of a focus from one vertex of the transverse axis is a finite line, and the eccentricity is infinitely great, that is, greater than any assignable line.*

The same conclusions may be drawn from the equations E_5 and H_5 of the rectangular co-ordinates of the ellipse and hyperbola; for, supposing $2a$ indefinitely great, while x has a finite magnitude, these become simply $y^2 = px$, the equation of a parabola.

Kepler was the first who introduced the notion of infinity into geometry: it was indeed a bold step in the progress of the

science, but it has led to most of the fine discoveries which have been made since his time.

When, in the equations to rectangular co-ordinates, the origin is at the pole, the equations may be readily changed to polar equations; for then $x = r \cos. v$ and $y = r \sin. v$; therefore we have now only to substitute for x and y in the former their values, and the results will be polar equations.

Thus, supposing the centres of the ellipse and hyperbola to be the origin of co-ordinates, and the radius vector to be a semidiameter, and u to be the angle which it makes with the transverse axis, from equations (E_2) and (H_2) we find

$$r^2 = \frac{a^2 b^2}{a^2 \sin.^2 u + b^2 \cos.^2 u} = \frac{a^2 b^2}{a^2 - e^2 \cos.^2 u} \dots\dots\dots\dots\dots\dots (E_8),$$

$$r^2 = \frac{a^2 b^2}{b^2 \cos.^2 u - a^2 \sin.^2 u} = \frac{a^2 b^2}{e^2 \cos.^2 u - a^2} \dots\dots\dots\dots\dots\dots (H_8).$$

These are the equations of the ellipse and hyperbola, the centres being the poles. In this way all the other polar equations of the curves might have been found; and, by a reverse process, the equations to rectangular co-ordinates may be deduced from the polar.

SECTION III.

ANALYTICAL INVESTIGATION OF SERIES FOR THE AREAS OF THE CIRCLE AND EQUILATERAL HYPERBOLA.

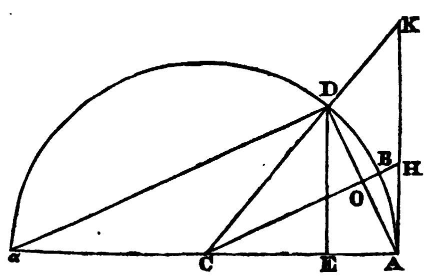
1. The quadrature of the parabola was perfectly accomplished by Archimedes: but it has been found impossible to find rectilinear spaces exactly equal to elliptic and hyperbolic areas; these can only be exhibited numerically by infinite series. In what follows, series will be investigated from principles as elementary as the nature of the subject will admit of; but it will be necessary to employ the symbols of algebra in the reasoning; and the investigation will be confined to sectors of a circle and equilateral hyperbola, from which sectors of any ellipse and hyperbola may be found.

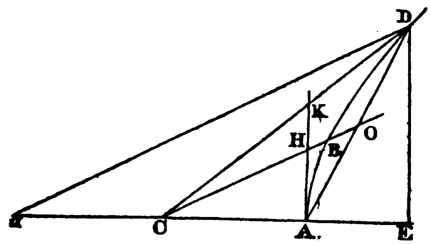
The following geometrical proposition is to be premised.

THEOREM.

Let C *be the centre of a circle, or of an equilateral hyperbola, and* CA *the semitransverse axis; let* ACB, BCD, *be two equal sectors of the circle or hyperbola, and let* AHK, *a tangent at the vertex* A, *meet the semidiameters* CB, CD *in* H *and* K.

$$\text{In the circle, } 2CA : AK = CA^2 - AH^2 : CA \cdot AH,$$
$$\text{In the hyperbola, } 2CA : AK = CA^2 + AH^2 : CA \cdot AH.$$





In each curve, draw DA, Da to the extremities of the axis; let the chord AD meet the semidiameter CB in O, and draw DE perpendicular to the axis.

Because AO = OD (3, 3, E. and Prop. 33, Part III.), and AC = Ca, the lines CH, aD are parallel (2, 6, E.), therefore the triangles CAH, aED are similar.

Now $ED^2 = AE \cdot Ea$ (35, 3, E. and 21 of Part III.),

therefore $AE : ED = ED : Ea$ (16, 6, E.);

hence the triangles DEA, aED are similar (6, 6, E.), and each is similar to CAH, therefore

$$aE : ED = AC : AH = AC^2 : AC \cdot AH;$$

and $AE : ED = AH : AC = AH^2 : AC \cdot AH$.

Hence, in the circle,

$$aE - AE \text{ or } 2CE : ED = AC^2 - AH^2 : AC \cdot AH,$$

and in the hyperbola (24, 5, E.),

$$aE + AE \text{ or } 2CE : ED = AC^2 + AH^2 : AC \cdot AH.$$

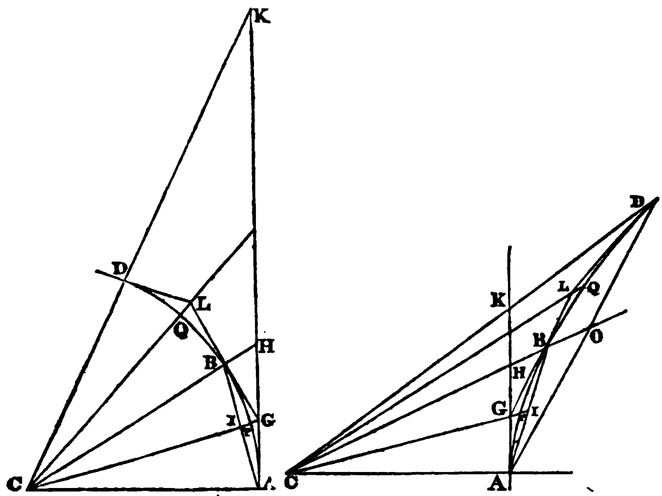
But $2CE : ED = 2CA : AK$;

therefore, in the circle, $2CA : AK = AC^2 - AH^2 : AC \cdot AH$;

and, in the hyperbola, $2CA : AK = AC^2 + AH^2 : AC \cdot AH$.

PROBLEM I.

2. *To investigate formulæ that shall express the reciprocal of the area of any sector of a circle, or of an equilateral hyperbola.*



Let ACD be a sector of a circle or hyperbola, and AC the radius of the circle, or semitransverse axis of the hyperbola: let the sector be bisected by the semidiameter CB, and, again, each of the sectors ACB, BCD by the semidiameters CP, CQ, and so on; thus dividing the sector ACB first into two equal sectors, then into four, then into eight, and so on. Draw tangents to the curves at the alternate points A, B, D; and because, from the nature of the curves, the chords which join these points are ordinates to the semidiameters CP, CQ which pass between them, the tangents will intersect each other at G and L, points in the semidiameters (1 Cor. Prop. 10, Part III.). Join A, B, and let the chord AB meet CP in I; and because the triangles ACI, BCI are equal (38, 1, E.), and also the triangles AGI, BGI, the triangles CAG, CBG are equal. In the same way it appears that the triangles DCL, BCL are equal.

Again, because the semidiameter CB bisects the chord which joins the points P, Q, it will bisect GL, the tangent parallel to the chord; therefore the triangles GCB, LCB are equal. Hence

it appears that the triangles ACG, GCB, BCL, LCD are all equal; and this will be true whatever be their number.

3. Let AK, a tangent at the vertex A, meet the semidiameters CP, CB, CD, in G, H, K. Put a to denote CA, the radius of the circle or semiaxis of the hyperbola, and let the lines AK, AH, AG, &c. be denoted by t, t', t'', t''', t^{IV}, &c. These lines in the circle are the tangents of the arcs AD, AB, AP, &c. We shall, by analogy, consider them as *tangents* corresponding to the sectors DCA, BCA, PCA, &c. in the two curves.

In the figure, the polygon contained by the semidiameters CA, CD, and the tangents AG, GL, LD, is made up of four triangles, each equal to the triangle ACG; but if each fourth of the sector were again bisected, and tangents drawn at the extremities of the diameters, there would be formed a polygon made up of eight equal triangles; each repetition of bisection doubling the number of triangles. Suppose that in this way a polygon has been formed composed of sixteen equal triangles. The area of the triangle next the semiaxis CA will be $\frac{1}{2}\,at^{\text{IV}}$; and, in this case, if P denote the area of the polygon so formed, $2\text{P} = 16at^{\text{IV}}$.

Now, by art. 1, $2a : t = a^2 \mp t'^2 : at'$; therefore, $a^2t \mp t'^2\,t = 2a^2t'$; hence the following identical equations are formed,

$$\frac{a}{t} = \frac{a}{2t'} \mp \frac{t'}{2a},$$

$$\frac{a}{2t'} = \frac{a}{4t''} \mp \frac{t''}{4a},$$

$$\frac{a}{4t''} = \frac{a}{8t'''} \mp \frac{t'''}{8a},$$

$$\frac{a}{8t'''} = \frac{a}{16t^{\text{IV}}} \mp \frac{t^{\text{IV}}}{16a}.$$

By adding these, and rejecting terms common to both sides of the result, and putting 2P instead of $16at^{\text{IV}}$, we obtain

$$\frac{a}{t} = \frac{a^2}{2\text{P}} \mp \left\{ \frac{t'}{2a} + \frac{t''}{4a} + \frac{t'''}{8a} + \frac{t^{\text{IV}}}{16a} \right\}$$

and from this again,

$$\frac{a^3}{2\mathrm{P}} = \frac{a^2}{t} \pm \left\{\tfrac{1}{2}t' + \tfrac{1}{4}t'' + \tfrac{1}{8}t''' + \tfrac{1}{16}t^{\mathrm{IV}}\right\}.$$

This is a particular case, but the property is general, and is an elegant geometrical theorem, not commonly known, which may be enunciated as follows:

Theorem.

If a sector of a circle, or an equilateral hyperbola, between its semiaxis and any semidiameter, be divided by semidiameters into 2^n equal parts (n being any number), and straight lines be drawn touching the curve at the vertices of the extreme and intermediate semidiameters; then, putting P *for the area of the polygon contained by the extreme semidiameters and the lines which touch the curve, and t for the tangent of the whole sector, t' for that of its half, t'' for that of its fourth, and so on to $t^{(n)}$, the last of the series (which will be the tangent of the sector next the semiaxis), and a for the semiaxis; we have*

$$\frac{a^3}{2\mathrm{P}} = \frac{a^2}{t} \pm \left\{\frac{1}{2}t' + \frac{1}{4}t'' + \frac{1}{8}t''' \ldots\ldots + \frac{1}{2^n}t^{(n)}\right\} *$$

the upper part of the sign $\pm$ applying to the circle, and the lower to the hyperbola.

By making 2^n equal to 2, 4, 8, 16, &c. successively, we may have a polygon which shall differ from the sector by less than any assignable space. If we suppose 2^n indefinitely great, then the polygon may be considered as equal to the sector; so that, denoting the sector by s,

$$\frac{a^3}{2s} = \frac{a^2}{t} \pm \left\{\tfrac{1}{2}t' + \tfrac{1}{4}t'' + \tfrac{1}{8}t''' + \tfrac{1}{16}t^{\mathrm{IV}} + \&c.\right\} \ldots\ldots (\mathrm{A}),$$

* The above theorem, and others of a like kind, were given by the author of this treatise in a Memoir read in the Royal Society of Edinburgh in June 1808.

the series being continued indefinitely: this is a solution of the problem.

This formula will still manifestly hold true if $\frac{1}{2}s$ be put for s, provided t' be put for t, t'' for t', &c.; it then becomes

$$\frac{a^3}{s} = \frac{a^2}{t'} \pm \left\{ \frac{t''}{2} + \frac{t'''}{4} + \frac{t^{\text{IV}}}{8} + \text{\&c.} \right\} \text{.........}(A').$$

The upper part of the sign $\pm$ applies to the circle, the lower to the hyperbola.

4. In applying the formula, the values of the tangents AH $= t'$, AG $= t''$, &c. must be found from the first t and from one another.

In the circle we have found (art. 3) that

$$\frac{a}{t'} - \frac{t'}{a} = \frac{2a}{t} \text{.....................}(1).$$

By taking the squares of these equals, then adding 4 to the results, and taking the square roots of these last, we have

$$\frac{a}{t'} + \frac{t'}{a} = \frac{2a}{t}\sqrt{\left\{1 + \frac{t^2}{a^2}\right\}} \text{.....................}(2).$$

By subtracting (1) from (2), we find

$$t' = \frac{a^2}{t}\left\{\sqrt{\left(1 + \frac{t^2}{a^2}\right)} - 1\right\}.$$

Similarly we may find

$$t'' = \frac{a^2}{t'}\left\{\sqrt{\left(1 + \frac{t'^2}{a^2}\right)} - 1\right\}, \text{ \&c.}$$

A few of the quantities t', t'', t''', &c. may be computed from the formula; but when one (t''', for instance) has been found such that $\frac{t'''}{a}$ is a small fraction, those which follow may be most readily computed by a series obtained from $\sqrt{\left(1 + \frac{t^2}{a^2}\right)}$ by evolution, or the binomial theorem; thus, since

$$\sqrt{\left(1 + \frac{t^2}{a^2}\right)} = 1 + \tfrac{1}{2}\frac{t^2}{a^2} - \tfrac{1}{8}\frac{t^4}{a^4} + \tfrac{1}{16}\frac{t^6}{a^6} - \text{, \&c.};$$

therefore $t' = \frac{1}{2}t - \frac{1}{8}\frac{t^3}{a^2} + \frac{1}{16}\frac{t^5}{a^4} -$, &c. $= \frac{t}{2} - \frac{1}{a^2}\left(\frac{t}{2}\right)^3$ nearly.

5. For the quadrature of the hyperbola, the tangents t', t'', &c. may be found from the first t by formulæ entirely similar to the above, differing only in the signs of the terms. In this curve, however, the tangents have a property which those in the circle have not, by which their computation may be facilitated. For since, in the hyperbola,

$$a : t = a^2 + t'^2 : 2at' \text{.......(art. 1)},$$

$$\text{therefore, } a + t : a - t = a^2 + 2at' + t'^2 : a^2 - 2at' + t'^2;$$

$$\text{that is, } a + t : a - t = (a + t')^2 : (a - t')^2.$$

$$\text{Hence } \frac{a + t'}{a - t'} = \left(\frac{a + t}{a - t}\right)^{\frac{1}{2}};$$

$$\text{and, similarly, } \frac{a + t''}{a - t''} = \left(\frac{a + t'}{a - t'}\right)^{\frac{1}{2}}, \text{ \&c.}$$

$$\text{Now, let } \frac{a + t}{a - t} = v^2; \text{ then } \frac{a + t'}{a - t'} = v; \ \frac{a + t''}{a - t''} = v^{\frac{1}{2}}, \text{ \&c.};$$

$$\text{and hence } \frac{t}{a} = \frac{v^2 - 1}{v^2 + 1}, \frac{t'}{a} = \frac{v - 1}{v + 1}, \frac{t''}{a} = \frac{v^{\frac{1}{2}} - 1}{v^{\frac{1}{2}} + 1}, \text{ \&c.}$$

By these formulæ, in the case of the hyperbola, all the tangents t', t'', &c. may be readily found from the first.

6. From the preceding investigation, we have the following formulæ for the areas of sectors of the two curves.

I.—THE CIRCLE.

Let s = the sector ACD, a = AC the radius, φ = angle ACD. Putting instead of t, t', t'', &c. their values a tan. φ, a tan. $\frac{1}{2}\varphi$, a tan. $\frac{1}{4}\varphi$, &c. in the series (A), we have

$$(C_1)$$

$$\frac{1}{s} = \frac{2}{a^2}\left\{\cot.\varphi + (\tfrac{1}{2}\tan.\tfrac{1}{2}\varphi + \tfrac{1}{4}\tan.\tfrac{1}{4}\varphi + \tfrac{1}{8}\tan.\tfrac{1}{8}\varphi +, \text{ \&c.}\right\}$$

II.—THE HYPERBOLA.

The values of $\frac{t}{a}$, $\frac{t'}{a}$, $\frac{t''}{a}$, &c. found, as directed, in art. 5, being substituted in formulæ (A) and (A′), they become

$$(H_1)$$

$$\frac{a^2}{2s} = \frac{a}{t} - \left\{ \frac{1}{2}\frac{v-1}{v+1} + \frac{1}{4}\frac{v^{\frac{1}{2}}-1}{v^{\frac{1}{2}}+1} + \frac{1}{8}\frac{v^{\frac{1}{4}}-1}{v^{\frac{1}{4}}+1}, \text{ \&c. } \right\}$$

$$(H'_1)$$

$$\frac{a^2}{s} = \frac{v+1}{v-1} - \left\{ \frac{1}{2}\frac{v^{\frac{1}{2}}-1}{v^{\frac{1}{2}}+1} + \frac{1}{4}\frac{v^{\frac{1}{4}}-1}{v^{\frac{1}{4}}+1} + \frac{1}{8}\frac{v^{\frac{1}{8}}-1}{v^{\frac{1}{8}}+1}, \text{ \&c.} \right\}$$

7. *Example of the Application of the Series* (C_1).

To find the ratio of the diameter of a circle to its circumference.

Let this ratio be that of $1 : \pi$, so that when the radius $= 1$, the circumference $= 2\pi$, and the area $=$ rad. $\times$ $\frac{1}{2}$ circumference $= \pi$, and supposing s a quadrant, $s = \frac{\pi}{4}$. When $\varphi = 90°$, cot. $\varphi = 0$, and the series gives

$$\frac{2}{\pi} = \tfrac{1}{2} \tan. \tfrac{1}{2}\varphi + \tfrac{1}{4} \tan. \tfrac{1}{4}\varphi + \tfrac{1}{8} \tan. \tfrac{1}{8}\varphi +, \text{\&c.}$$

The tangents may be computed by the formulæ of art. 4, for finding t' from t, t'' from t', &c. which, by putting α for any angle, may be expressed thus:

$$\tan. \tfrac{1}{2}\alpha = \left\{ \frac{1}{\tan. \alpha}\sqrt{(1 + \tan.^2 \alpha)} - 1 \right\}$$

Putting now φ, $\frac{1}{2}\varphi$, $\frac{1}{4}\varphi$, $\frac{1}{8}\varphi$, &c. instead of α, and observing that $\tan. \frac{1}{2}\varphi = 1$, we find

Tan. $\frac{1}{2}\ \varphi = 1$		$\frac{1}{2}$ tan. $\frac{1}{2}\ \varphi = \cdot 50000000000$
Tan. $\frac{1}{4}\ \varphi =$	$\cdot 41421356237$	$\frac{1}{4}$ tan. $\frac{1}{4}\ \varphi = \cdot 10355339059$
Tan. $\frac{1}{8}\ \varphi =$	$\cdot 19891236738$	$\frac{1}{8}$ tan. $\frac{1}{8}\ \varphi = \cdot 02486404592$
Tan. $\frac{1}{16}\ \varphi =$	$\cdot 09849140336$	$\frac{1}{16}$ tan. $\frac{1}{16}\ \varphi = \cdot 00615571271$
Tan. $\frac{1}{32}\ \varphi =$	$\cdot 04912684977$	$\frac{1}{32}$ tan. $\frac{1}{32}\ \varphi = \cdot 00153521406$
Tan. $\frac{1}{64}\ \varphi =$	$\cdot 02454862211$	$\frac{1}{64}$ tan. $\frac{1}{64}\ \varphi = \cdot 00038357222$
Tan. $\frac{1}{128}\ \varphi =$	$\cdot 01227246238$	$\frac{1}{128}$ tan. $\frac{1}{128}\ \varphi = \cdot 00009587861$
Tan. $\frac{1}{256}\ \varphi =$	$\cdot 00613600016$	$\frac{1}{256}$ tan. $\frac{1}{256}\ \varphi = \cdot 00002396875$
Tan. $\frac{1}{512}\ \varphi =$	$\cdot 00306797120$	$\frac{1}{512}$ tan. $\frac{1}{512}\ \varphi = \cdot 0000059921$
		$\frac{1}{3}$ of the preceding term $= \cdot 00000199738$

$$\frac{2}{\pi} = \cdot 63661977237$$
$$\pi = 3\cdot 1415926536$$

This number, a most important element in geometry, is true to the last figure.

Problem II.

8. *To investigate formulæ which shall express the second power of the area of a circle, or equilateral hyperbola.*

Let the letters a, t, t', t'', t''', t^{IV}, P, s denote the same things as in Problem I. It was found that $\frac{a}{t} \mp \frac{t'}{a} = \frac{2a}{t}$; from this, by taking the squares of the two equals, we have $\frac{a^2}{t'^2} + \frac{t'^2}{a^2} \mp 2 = \frac{4a^2}{t^2}$. Hence there is obtained the following series of identical equations.

$$\frac{a^2}{t^2} = \frac{a^2}{2^2 t'^2} + \frac{1}{4}\,\frac{t'^2}{a^2} \mp \frac{1}{2},$$
$$\frac{a^2}{2^2 t'^2} = \frac{a^2}{4^2 t''^2} + \frac{1}{4^2}\,\frac{t''^2}{a^2} \mp \frac{1}{2\cdot 4},$$

$$\frac{a^2}{4^2 t''^2} = \frac{a^2}{8^2 t'''^2} + \frac{1}{4^3}\frac{t''^2}{a^2} \mp \frac{1}{2 \cdot 4^2},$$

$$\frac{a^2}{8^2 t'''^2} = \frac{a^2}{16^2 t^{\mathrm{IV}2}} + \frac{1}{4^4}\frac{t^{\mathrm{IV}2}}{a^2} \mp \frac{1}{2 \cdot 4^3},$$

which may be continued to any extent. By adding these and rejecting terms common to both sides, and putting $(2\mathrm{P})^2$ for its equal $(16at^{\mathrm{IV}})^2$, we have

$$\frac{a^2}{t^2} = \begin{cases} \dfrac{a^4}{4\mathrm{P}^2} \mp \dfrac{1}{2}\left(1 + \dfrac{1}{4} + \dfrac{1}{4^2} + \dfrac{1}{4^3}\right) \\ + \dfrac{1}{4}\dfrac{t'^2}{a^2} + \dfrac{1}{4^2}\dfrac{t''^2}{a^2} + \dfrac{1}{4^3}\dfrac{t'''^2}{a^2} + \dfrac{1}{4^4}\dfrac{t^{\mathrm{IV}2}}{a^2}. \end{cases}$$

The sum of the geometrical series $1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3}$ is $\frac{4}{3}\left(1 - \frac{1}{4^4}\right)$. If now, instead of four terms, we suppose their number to be n, and put $t^{(n)}$ to denote the last tangent in the series t, t', t'', t''', &c., and $\frac{4}{3}\left(1 - \frac{1}{4^n}\right)$ for the geometrical series, and $\frac{1}{4^n}\left(\frac{t^{(n)}}{a}\right)^2$ for $\frac{1}{4^4}\left(\frac{t^{\mathrm{IV}}}{a}\right)^2$, we get, after transposing terms, the following proposition:

Theorem.

If a sector of a circle, or equilateral hyperbola, between its semiaxes and any semidiameter, be divided by semidiameters (as in the fig. of Prop. I.) into 2^n *equal parts (n being any number), and straight lines be drawn touching the curve at the vertices of the extreme and intermediate semidiameters; then, putting* P *for the area of the polygon contained by the extreme semidiameters and the lines which touch the curve, and t for the tangent of the whole sector, t′ for that of its half, t″ for that of its fourth, and*

so on, and $t^{(n)}$ for the tangent of the last in the series t, t', &c. (which will be the tangent of the sector next the semiaxis), and a for the semiaxis, we have in every case

$$\frac{a^6}{4P^2}=\begin{cases}\dfrac{a^4}{t^2}\pm\dfrac{2}{3}a^2\left(1-\dfrac{1}{4^n}\right)\\ -\left\{\dfrac{1}{4}t'^2+\dfrac{1}{4^2}t''^2+\dfrac{1}{4^3}t'''^2\dots\dots\dots\dots+\dfrac{1}{4^n}\left(t^{(n)}\right)^2,\right.\end{cases}$$

the upper part of the sign $\pm$ applying to the circle, and the lower to the hyperbola.

This is a second elegant geometrical property of a polygon described about a sector of a circle or hyperbola.

If n be increased continually, the area of the polygon will approach to s, the area of the sector, which is its limit; therefore, supposing n to be indefinitely great, we have

$$\frac{a^6}{4s^2}=\frac{a^4}{t^2}\pm\frac{2}{3}a^2-\left\{\frac{t'^2}{4}+\frac{t''^2}{4^2}+\frac{t'''^2}{4^3}+\text{\&c.}\right\}\dots\dots\dots\text{(B)}$$

the upper part of the sign $\pm$ applying to the circle, and the lower to the hyperbola, and the series being continued indefinitely.

From the nature of the expression, we may evidently change s into $\frac{1}{2}s$, provided we also change t into t', and t' into t'', &c. Hence it follows that

$$\frac{a^6}{s^2}=\frac{a^4}{t'^2}\pm\frac{2}{3}a^2-\left\{\frac{t''^2}{4}+\frac{t'''^2}{4^2}+\frac{t^{\mathrm{IV}2}}{4^3}+\text{\&c.}\right\}\dots\dots\dots\text{(B}'\text{)}$$

9. To determine the quantities t'^2, t''^2, &c. it has been found (art. 4) that in the case of the circle,

$$\frac{tt'}{a}=\sqrt{(a^2+t^2)}-a\dots\dots\dots\dots\dots\dots\dots\text{(1)}.$$

$$\text{Now }\sqrt{(a^2+t^2)}-a=\frac{t^2}{\sqrt{(a^2+t^2)}+a};$$

$$\text{therefore }\frac{t'}{ta}=\frac{1}{\sqrt{(a^2+t^2)}+a}\dots\dots\dots\dots\text{(2)}.$$

hence, taking the product of (1) and (2),

$$\frac{t'^2}{a^2}=\frac{\sqrt{(a^2+t^2)}-a}{\sqrt{(a^2+t^2)}+a}\dots\dots\dots\dots(c).$$

This is for the circle, but by a like process we have for the hyperbola

$$\frac{t'^2}{a^2}=\frac{a-\sqrt{(a^2+t^2)}}{a+\sqrt{(a^2+t^2)}}\dots\dots\dots\dots(h).$$

By these the terms t''^2, t'''^2, &c. may be derived each from that before it.

The series (B) is alike applicable to the two curves, but its form may be modified by properties peculiar to each curve.

10. In the circle, supposing a the radius $=1$, and putting φ for DCA the angle of the sector, or the arc AD, then $t=\tan.\,\varphi$, $t'=\tan.\,\frac{1}{2}\varphi$, $t''=\tan.\,\frac{1}{4}\varphi$, &c. and from (c) the first of the two formulæ (art. 9) observing that $\sec.\,\varphi\cos.\,\varphi=1$

$$t'^2=\frac{\sec.\varphi-1}{\sec.\varphi+1}=\frac{1-\cos.\varphi}{1+\cos.\varphi};\ t''^2=\frac{1-\cos.\frac{1}{2}\varphi}{1+\cos.\frac{1}{2}\varphi},\ \&c.$$

Considering now that $s=\frac{1}{2}\text{ rad.}\times\varphi=\frac{1}{2}\varphi$, we have (by B′) the following formula,

(C_2)

To find the length of φ any arc of a circle.

$$\frac{1}{\varphi^2}=\left\{\begin{array}{l}\frac{1}{4}\cdot\frac{1+\cos.\varphi}{1-\cos.\varphi}+\frac{1}{6}\\ \qquad-\left\{\frac{1}{4^2}\cdot\frac{1-\cos.\frac{1}{2}\varphi}{1+\cos.\frac{1}{2}\varphi}+\frac{1}{4^3}\cdot\frac{1-\cos.\frac{1}{4}\varphi}{1+\cos.\frac{1}{4}\varphi}\right.\\ \qquad\qquad\left.+\frac{1}{4^4}\cdot\frac{1-\cos.\frac{1}{8}\varphi}{1+\cos.\frac{1}{8}\varphi}+\&^c.\right\}\end{array}\right.$$

The cosines may be derived each from that before it by a known formula, viz.

$$\text{Cos.}\frac{1}{2}\varphi=\sqrt{\frac{1+\cos.\varphi}{2}},\ \cos.\frac{1}{4}\varphi=\sqrt{\frac{1+\cos.\frac{1}{2}\varphi}{2}}.$$

11. To find a similar expression for the hyperbolic sector; put x for its absciss, y for its ordinate, and t for its tangent; and

similarly x', y', and t' for the absciss, ordinate, and tangent of its half.

We have found (art. 5) that $\left(\frac{a+t'}{a-t'}\right)^2 = \frac{a+t}{a-t}$;

Now by similar triangles $\frac{t}{a} = \frac{y}{x}$, therefore $\frac{a+t}{a-t} = \frac{x+y}{x-y}$ $= \frac{(x+y)^2}{x^2-y^2} = \frac{(x+y)^2}{a^2}$, and similarly $\frac{a+t'}{a-t'} = \frac{(x'+y')^2}{a^2}$;

$$\text{therefore } \left(\frac{x'+y'}{a}\right)^2 = \frac{x+y}{a} \quad\text{..................(1)};$$

from which it follows that $\left(\frac{x'-y'}{a}\right)^2 = \frac{x-y}{a}$ ....(2),

as will appear by multiplying the corresponding sides of the two equations.

By adding (1) and (2), we find $x'^2 + y'^2 = ax$.

Now $x'^2 - y'^2 = a^2$;

therefore, $2x'^2 = a\,(x+a)$, and $2y'^2 = a\,(x-a)$;

hence, observing that $\frac{t}{a} = \frac{y'}{x'}$, we have $\frac{t'^2}{a^2} = \frac{y'^2}{x'^2} = \frac{x-a}{x+a}$.

Exactly in the same way we may find

$$\frac{t''^2}{a^2} = \frac{x'-a}{x'+a},\quad \frac{t'''^2}{a^2} = \frac{x''-a}{x''+a}, \text{ \&c.}$$

Again, since $2x'^2 = a\,(x+a)$, we have similarly

$$2x''^2 = a\,(x'+a),\ 2x'''^2 = a\,(x''+a), \text{ \&c.}$$

The result of the analysis in the case of the hyperbola may be stated as follows.

12. *To find the area of a sector of an equilateral hyperbola.* (H_2).

Let s denote the area of the sector ACD, a the semiaxis, and x its absciss. Compute the series of quantities x', x'', x''', &c. from these formulæ;

$$x' = \sqrt{\frac{a\,(x+a)}{2}},\ x'' = \sqrt{\frac{a\,(x'+a)}{2}},\ x''' = \sqrt{\frac{a\,(x''+a)}{2}}, \text{ \&c.}$$

Then, from the formula (B′) we have

$$\frac{a^4}{s^2} = \left\{ \begin{array}{l} \dfrac{x+a}{x-a} - \dfrac{2}{3} \\ -\left\{ \dfrac{1}{4} \cdot \dfrac{x'-a}{x'+a} + \dfrac{1}{4^2} \cdot \dfrac{x''-a}{x''+a} + \dfrac{1}{4^3} \cdot \dfrac{x'''-a}{x'''+a} + \&c. \right\}. \end{array} \right.$$

The terms of this series, like that for the circle, approach to those of a geometrical series, of which the ratio is one sixteenth; so that when a term has been found nearly one fourth of that before it, the sum of those that follow will be nearly one fifteenth of that term.

The trigonometrical tables apply elegantly to the general formula (B) in the case of the hyperbola, by the analytical artifice of subsidiary angles.

Let 2φ be any angle, because $1 = \cos.^2 \varphi + \sin.^2 \varphi$, and $\sin. 2\varphi = 2 \sin. \varphi \cos. \varphi$; therefore,

$1 + \sin. 2\varphi = \cos.^2 \varphi + 2 \cos. \varphi \sin. \varphi + \sin.^2 \varphi = (\cos. \varphi + \sin. \varphi)^2$

$1 - \sin. 2\varphi = \cos.^2 \varphi - 2 \cos. \varphi \sin. \varphi + \sin.^2 \varphi = (\cos. \varphi - \sin. \varphi)^2$

Now $(\cos. \varphi + \sin. \varphi)^2 = \cos.^2 \varphi \, (1 + \tan. \varphi)^2$,

and $(\cos. \varphi - \sin. \varphi)^2 = \cos.^2 \varphi \, (1 - \tan. \varphi)^2$;

therefore, $\dfrac{1 - \sin. 2\varphi}{1 + \sin. 2\varphi} = \left(\dfrac{1 - \tan. \varphi}{1 + \tan. \varphi}\right)^2$.

We have found that in the hyperbola, $\dfrac{1 - \dfrac{t}{a}}{1 + \dfrac{t}{a}} = \left\{ \dfrac{1 - \dfrac{t'}{a}}{1 + \dfrac{t'}{a}} \right\}^2$;

this expression, compared with the angular formula, shows that t is related to t' exactly as the sine of an angle is to the tangent of its half; so that if we assume $\dfrac{t}{a} = \sin. \theta$, then $\dfrac{t'}{a} = \tan. \dfrac{1}{2} \theta$;

and if $\sin. \theta' = \tan. \frac{1}{2} \theta$, from $\dfrac{t'}{a} = \sin. \theta'$ we have $\dfrac{t''}{a} = \tan. \frac{1}{2} \theta'$, and so on.

Now, x and y being the co-ordinates of an hyperbolic sector, $\dfrac{t}{a} = \dfrac{y}{x}$; and $\sin. \theta = \dfrac{\tan. \theta}{\sec. \theta}$; therefore $\dfrac{y}{x} = \dfrac{\tan. \theta}{\sec. \theta}$. This equa-

tion is satisfied by making $\frac{x}{a} = \sec.\theta$ and $\frac{y}{a} = \tan.\ \theta$, for then $\frac{x^2 - y^2}{a^2} = 1$. Let C denote the angle of the sector; then $\tan.\ C = \frac{y}{x} = \frac{\tan.\theta}{\sec.\theta} = \sin.\ \theta$.

From the preceding analysis we obtain by substitution in formula (B), and observing that φ being any angle, $2\sin.^2\varphi = 1 - \cos.2\varphi$, the following expressions. As usual, let a be the semiaxis of an equilateral hyperbola, s a sector, x and y its coordinates, t its tangent, C the angle of the sector; to find s, the area, having given any one of the quantities x, y, t, C.

1. Find an angle θ from one of these equations, $\cos.\ \theta = \frac{a}{x}$; $\tan.\ \theta = \frac{y}{a}$; $\sin.\ \theta = \frac{t}{a} = \tan.\ C$.

2. Find a series of angles θ', θ'', θ''', &c. such that $\sin.\ \theta' = \tan.\frac{1}{2}\theta$, $\sin.\ \theta'' = \tan.\frac{1}{2}\theta'$, $\sin.\ \theta''' = \tan.\frac{1}{2}\theta''$, &c.

3. The area s will be expressed by either of these formulæ;

$$(H'''_2)$$

$$\frac{a^4}{s^2} = \frac{4}{\sin.^2\theta} - \frac{8}{3} - \left\{ \sin.^2\theta' + \frac{1}{4}\sin.^2\theta'' + \frac{1}{4^2}\sin.^2\theta''' +, \text{\&c.} \right\},$$

$$(H^{\text{iv}}_2)$$

$$\frac{a^4}{s^2} = \begin{cases} \dfrac{8}{1 - \cos.2\theta} - \dfrac{8}{3} \\ -\left\{ \frac{1}{2}(1 - \cos.2\theta') + \frac{1}{8}(1 - \cos.2\theta'') + \frac{1}{32}(1 - \cos.2\theta''') +, \right. \\ \qquad \left. \text{\&c.} \right\} \end{cases}$$

15. The following examples will show the application of these formulæ.

1. To find the ratio $1 : \pi$, viz. the ratio of the diameter of a circle to its circumference from the series (C_2).

In this case, making $\varphi = 90°$, we have

$$\cos.\varphi=0 \qquad \frac{1}{4}\cdot\frac{1+\cos.\varphi}{1-\cos.\varphi}+\frac{1}{6}=\frac{5}{12}=\cdot416666667$$

$$\cos.\tfrac{1}{2}\varphi=0\cdot70710678 \qquad \frac{1}{4^2}\cdot\frac{1-\cos.\frac{1}{2}\varphi}{1+\cos.\frac{1}{2}\varphi}=\cdot010723305$$

$$\cos.\tfrac{1}{4}\varphi=0\cdot92387953 \qquad \frac{1}{4^3}\cdot\frac{1-\cos.\frac{1}{4}\varphi}{1+\cos.\frac{1}{4}\varphi}=\cdot000618221$$

$$\cos.\tfrac{1}{8}\varphi=0\cdot98078528 \qquad \frac{1}{4^4}\cdot\frac{1-\cos.\frac{1}{8}\varphi}{1+\cos.\frac{1}{8}\varphi}=\cdot000037893$$

$$\tfrac{1}{15}\text{ of the preceding term}=\cdot000002526$$

$$\cdot011381945$$

$$\frac{1}{\varphi^2}=\cdot405284722$$

$$\varphi=1\cdot5707964$$

$$\pi=2\varphi=3\cdot1415927$$

2. To find s, the area of a sector of an equilateral hyperbola, supposing a the semiaxis $=1$, and x the absciss of the sector $=1\cdot25$.

Proceeding according to the formula (H_2),

$$\frac{x+1}{x-1}-\tfrac{2}{3}=9-\tfrac{2}{3}=\tfrac{25}{3}=8\cdot33333333$$

$$x'=1\cdot06066017 \qquad \frac{1}{4}\cdot\frac{x'-1}{x'+1}=0\cdot00735931$$

$$x''=1\cdot01505176 \qquad \frac{1}{4^2}\cdot\frac{x''-1}{x''+1}=0\cdot00046685$$

$$x'''=1\cdot00375589 \qquad \frac{1}{4^3}\cdot\frac{x'''-1}{x'''+1}=0\cdot00002929$$

$$\tfrac{1}{15}\text{ of the preceding term}=0\cdot00000195$$

$$0\cdot00785740$$

$$\frac{1}{s^2}=8\cdot32547593$$

$$s=0\cdot3465736$$

3. Taking the same example, to find s by the angular formula (H_2^{IV}).

Because $a=1$, $x=1\cdot25$, therefore $\cos.\,\theta=\dfrac{a}{x}=\dfrac{4}{5}$, $\sin.\,\theta=\dfrac{3}{5}$

$2 \sin.^2 \theta = 1 - \cos. 2\theta = \frac{18}{25}$: In all cases $\tan.^2 \frac{1}{2}\theta = \frac{1 - \cos.\theta}{1 + \cos.\theta} = \frac{x - a}{x + a}$, in the present case $= \frac{1}{9}$. Proceeding now according to the formula, we have, from Hutton's Table of Natural Sines and Tangents,

$$\tan. \tfrac{1}{2}\theta = \tfrac{1}{3} \qquad = \cdot 3333333 = \sin. (\theta' = 19^\circ\ 28' \cdot 28)$$
$$\tan. (\tfrac{1}{2}\theta' = 9^\circ 44' \cdot 14) = \cdot 1715739 = \sin. (\theta'' = 9\ \ 52 \cdot 76)$$
$$\tan. (\tfrac{1}{2}\theta'' = 4\ 56 \cdot 38) = \cdot 0864277 = \sin. (\theta''' = 4\ \ 57 \cdot 5)$$
$$\tan. (\tfrac{1}{2}\theta''' = 2\ 28 \cdot 75) = \cdot 0432966 = \sin. (\theta^{\text{iv}} = 2\ \ 28 \cdot 8).$$

Having found an angle θ^{iv} to be $\frac{1}{2}$ of that before it, the same would be true of all that follow; it is therefore needless to go farther. Indeed θ^{iv} is not wanted.

$$\frac{4}{\sin.^2\theta} - \frac{8}{3} = \frac{76}{9} = 8 \cdot 4444444$$

$$\cos. 2\theta' = 1 - 2 \sin.^2\theta' = \tfrac{7}{9} \qquad \tfrac{1}{2}(1 - \cos. 2\theta') = \cdot 1111111$$
$$\cos. (2\theta'' = 19^\circ\ 45' \cdot 52) = \cdot 9411149 \quad \tfrac{1}{8}(1 - \cos. 2\theta'') = \cdot 0073594$$
$$\cos. (2\theta''' = 9\ \ 55\ \ ) = \cdot 9850593 \quad \tfrac{1}{32}(1 - \cos. 2\theta''') = \cdot 0004669$$
$$\tfrac{1}{15} \text{ of preceding term} = \cdot 0000311$$
$$\cdot 1189685$$

$$\frac{1}{s^2} = 8 \cdot 3254759$$
$$s = \cdot 3465736$$

This value of s is correct in all its figures.

THE END.

EDINBURGH:
Printed by THOMAS ALLAN & Co.
265 High Street.

A

CATALOGUE OF WORKS

PUBLISHED BY

ADAM AND CHARLES BLACK, EDINBURGH.

CLASSIFIED INDEX.

EDINBURGH : PRINTED BY STARK AND COMPANY, OLD ASSEMBLY CLOSE.

CLASSIFIED INDEX.

HISTORY.

LAW.

MEDICINE AND SURGERY.

MISCELLANEOUS.

NATURAL HISTORY.

POETRY AND FICTION.

POLITICAL ECONOMY.

RELIGIOUS AND MORAL WORKS.

RURAL AFFAIRS AND RURAL SPORTS.

TRAVELS.

CATALOGUE OF WORKS

PUBLISHED BY

ADAM AND CHARLES BLACK.

ALEXANDER (W. L.)—ANGLO-CATHOLICISM NOT APOSTOLICAL. Being an Enquiry into the Scriptural Authority of the Leading Doctrines advocated in "The Tracts for the Times" and other Publications of the Anglo-Catholic School. By W. Lindsay Alexander, M. A. 8vo, 8s. cloth.

"Mr Lindsay Alexander's learned and able work."—EDINBURGH REVIEW.

ALEXANDER (W. L.)—THE UNITY OF THE CHRISTIAN CHURCH, AND THE COMMUNION OF CHRISTIANS. A Discourse to which are prefixed Strictures on an Address to Dissenters, recently issued by the Scottish Central Board for vindicating the right of Dissenters. By William Lindsay Alexander, M. A. 8vo, 6d. sewed.

ALEXANDER (W. L.)—MEMOIRS OF THE REV. JOHN WATSON, Late Pastor of the Congregational Church, Musselburgh, and First Secretary to the Congregational Union for Scotland; with Notices of the early History of Congregationalism in Scotland, and of the Formation and Progress of the Congregational Union. By W. Lindsay Alexander, M. A. 12mo, with a Portrait. 3s. 6d. boards.

ALEXANDER.—AN ABRIDGEMENT OF THE ACTS OF THE PARLIAMENTS of SCOTLAND from the Reign of James I. in 1424 to the Union in 1707, containing verbatim the whole Scots Acts now in force and use, with Notes and References. By William Alexander, W. S. (By recommendation of the Court.) Royal 8vo, 21s. boards.

ALEXANDER.—A DIGEST OF THE NEW BANKRUPT ACT FOR SCOTLAND, with a copious Appendix, containing the Act itself, Practical Forms, &c. Intended for the use of Merchants and Bankers, as well as for that of the Legal Profession. By William Alexander, W. S. Second Edition, improved and enlarged. Royal 8vo, 18s. boards.

ALEXANDER.—AN ABRIDGEMENT OF THE ACTS OF SEDERUNT OF the LORDS of COUNCIL and SESSION, from the Institution of the Court in 1532 to the present Time, including verbatim the whole Acts now in force relating to Sheriff, Burgh, and Commissary Courts, Sheriff-Clerks' Fees, Procurators' Fees, &c. With Notes and References. By William Alexander, W. S. (With the special approbation of the Court and Dean of Faculty.) Royal 8vo, 18s. boards.

ALEXANDER.—SUPPLEMENT TO ABRIDGEMENT OF THE ACTS OF SEDERUNT; containing the Acts from 12th July 1837, to 24th December 1842, with Notes and References. By William Alexander, W. S. Royal 8vo, 7s. 6d. boards.

"The Court approve entirely of the proposed work, not doubting it will be prepared with the same care and accuracy as formerly."—LORD PRESIDENT'S CLERK'S LETTER.

ALEXANDER.—CHART OF THE PROCEEDINGS IN A SEQUESTRATION under the Act 2d and 3d Victoria, Cap. 41; with SUPPLEMENT, containing Forms of Affidavits by Creditors, Minutes of Meetings, the Act itself, and a copious Index. By William Alexander, W. S. 12mo, 6s. cloth. The Chart may be had separately, Cased 4s., or in Sheet 2s. 6d.

ALEXANDER.—ANALYSIS OF THE HERITABLE SECURITIES' ACT AND INFEFTMENT Act (now passed) with an Appendix containing the Acts themselves, and Practical Forms of the Writs and Instruments thereby introduced. By William Alexander, W. S. Royal 8vo, 5s. boards.

ARAGO.—LIFE OF JAMES WATT.
With Illustrative Notes; to which are subjoined Memoir on Machinery, considered in relation to the Prosperity of the Working Classes; Lord Jeffrey's Panegyric on Watt; and Lord Brougham's Historical Account of the Discovery of the Composition of Water. By M. Arago, Perpetual Secretary of the French Academy of Sciences. Second Edition. 8vo, with numerous Illustrations, 4s. 6d. cloth.

"No man of science ought to be without it."—ATLAS.

AUDUBON.—ORNITHOLOGICAL BIOGRAPHY,
Or an Account of the Habits of the Birds of the United States of America; accompanied by Descriptions of the objects represented in the work entitled, "The Birds of America," and interspersed with delineations of American Scenery and Manners. By John James Audubon, F. R. SS. L. and E., &c. &c. Illustrated with Engravings on Wood. 5 volumes, Royal 8vo, L. 6, 5s., cloth.

AUDUBON.—A SYNOPSIS OF THE BIRDS OF NORTH AMERICA.
By John James Audubon, F. R. SS. L. and E., &c. &c. &c. Royal 8vo, 12s. cloth.

AYTOUN.—THE POEMS OF SIR ROBERT AYTOUN.

From a Manuscript in possession of the Editor, entitled, "The Poems of that worthy Gentleman, Sir Robert Aytoun, Knight, Secretary to Anna and Mary, Queens of Great Britain, &c. One volume, with a Memoir of the Author, and an Introduction to the Poetry of the Period, by Charles Roger, Esq. of Dunino. Illustrated with a Genealogical Tree of the Family of Aytoun of Kinaldie, a Monument of the Author in Westminster Abbey, and a Glossary of Scotticisms. Post 8vo, 6s. cloth.

AZARA.—NATURAL HISTORY OF THE QUADRUPEDS OF PARAGUAY

AND THE RIVER LA PLATA, translated from the Spanish of Don Felix de Azara, with a Memoir of the Author, a Physical Sketch of the Country, and numerous Notes. By W. Perceval Hunter, Esq. F.G.S. Z.S., &c., Member of the Geological Society of France. First vol. (all that is published) 8vo, with a Map of Paraguay and Buenos Ayres, 10s. 6d. cloth.

"Azara's Memoirs are invaluable. His descriptions are not only accurate but masterly."—Swainson.

"Don Felix de Azara a ecrit deux excellent ouvrages sur l'histoire naturelle du Paraguay."—Cuvier.

BALLINGALL.—OUTLINES OF MILITARY SURGERY.

By Sir George Ballingall, M.D., F.R.C.S.E., Surgeon to the Queen, Regius Professor of Military Surgery in the University of Edinburgh, &c. &c. Third Edition, 8vo, pp. 568, 14s. cloth.

"A work which ought to be in the hands of every naval and military medical officer."—British and Foreign Medical Review.

BARETTI.—ITALIAN AND ENGLISH DICTIONARY,

Accompanied by an Italian Grammar by Giuseppe Baretti. Eighth Edition, corrected and enlarged, by Charles Thomson. 2 vols. 8vo, 20s.

BELL.—INSTITUTES OF SURGERY,

Arranged in the order of the Lectures delivered in the University of Edinburgh. By Sir Charles Bell, K.G.H., F.R.SS. L. and E., M.D., Gott, Professor of Surgery in the University of Edinburgh, Consulting Surgeon to the Hospital, &c. &c. Two vols. post 8vo, 15s. cloth.

BENNETT.—ACCOUNT OF SOUTH AUSTRALIA,

HISTORICAL AND DESCRIPTIVE, founded on the experience of a Three Years' Residence in that Colony. By J. F. Bennett. Small 8vo, 2s. cloth.

BETHUNE.—POLITICAL ECONOMY

Explained and Enforced in a Series of Lectures. By Alexander Bethune, Labourer, Author of "Tales and Sketches of the Scottish Peasantry," and John Bethune, a Fifeshire Forester. 12mo. 4s. cloth.

BETHUNE.—THE SCOTTISH PEASANT'S FIRESIDE;

A Series of Tales and Sketches illustrating the Character of the Peasantry of Scotland. By Alexander Bethune, Labourer. 12mo, 4s. cloth.

Mr Bethune's former volume was thus noticed by The Athenæum;—"It is the perfect propriety of his taste, no less than the thorough intimacy with the unobtrusive subjects he treats of, which gives Mr Bethune's little book a great charm in our eyes." The work was reviewed in terms equally favourable by The Spectator, Tait's Magazine, Chambers' Journal, and other Periodicals.

BLACK'S GENERAL ATLAS OF THE WORLD.

Containing sixty-one Folio Maps, engraved on Steel, in the first style of the Art, by Sidney Hall, Hughes, and others, with Geographical Descriptions, Statistical Tables, and an Index of all the names occurring in the several Maps, amounting to 57,000, with their Latitude and Longitude, and the number of the Map in which they will be found. New Edition, with numerous improvements and additions, L. 2, 16s. strongly and elegantly half bound in morocco, with gilt leaves. This Atlas, already favourably known by the large circulation of the former Edition, is unsurpassed either in accuracy or beauty of execution, by any work of its class, while in cheapness it is altogether unrivalled.

"Large enough to be distinct, without being so large as to be unwieldy; it has all that any one can require for general use, and all that could be introduced, without making it too bulky or too expensive, and so counterbalance its principal intention."—Church of England Quarterly Rev.

BLACK'S SCHOOL ATLAS OF MODERN GEOGRAPHY.

An entirely new collection of Maps, drawn by W. Hughes, F.R.G.S., Professor of Geography in the College for Civil Engineers; and engraved on Steel in the first style of art; with an Index of all the names contained in the work, exhibiting the Latitude and Longitude of each, and a reference to the Map in which it may be found. The Maps of Royal Quarto size, and may be had bound, either in quarto at 9s. 6d., or 8vo, 9s.

BLACK'S PICTURESQUE TOURIST OF SCOTLAND.

Containing an accurate Travelling Map; Sixteen Engraved Charts of Roads, Railroads, and Interesting Localities (including Plans of Edinburgh and Glasgow); numerous Views of the Scenery on Wood and Steel; and a copious Itinerary. Fourth Edition, corrected and improved. In a handsome portable volume, 8s. 6d. cloth.

"As nearly as possible what a Guide-Book ought to be,—sensible, concise in its information, with that touch of poetry which is no less indispensable in such a haunted land than details of distances and historical facts, but which requires sound taste in its introduction."—Athenæum.

BLACK'S PICTURESQUE TOURIST OF ENGLAND AND WALES.

Containing a General Travelling Map, with the Roads and Railways distinctly laid down; besides Sections of the more important Districts on an enlarged scale, and Engraved Charts of Roads, Railroads, and Interesting Localities. In a portable volume, 10s. 6d. cloth.

"A carefully executed work, prettily illustrated, with useful Maps."—Athenæum.

BROWNE.—WHAT ASYLUMS WERE, ARE, AND OUGHT TO BE,
Being the substance of Five Lectures delivered before the Managers of the Montrose Royal Lunatic Asylum. By W. A. F. Browne, Surgeon, Medical Superintendent of the Montrose Royal Lunatic Asylum, formerly President of the Royal Medical Society, Edinburgh, &c. &c. Post 8vo, 5s. cloth.

BUCHANAN.—A COMPREHENSIVE ATLAS OF MODERN GEOGRAPHY,
Consisting of Thirty Maps, correctly engraved from new Drawings, containing the latest Discoveries, and embodying in each Map a great variety of Statistical and other details. By Robert Buchanan, Teacher of Mathematics, Geography, &c. Quarto, coloured, 18s. bound.

BUCHANANI (GEO).—PARAPHRASIS PSALMORUM DAVIDIS POETICA.
Cum Schematis Metrorum et Notis, accesserunt pauca miscellanea ex ceteris ejusdem carminibus. By A. and J. Dickinson. 12mo, 3s. bound.

BUSHNAN.—THE PHILOSOPHY OF INSTINCT AND REASON.
By J. Stevenson Bushnan, M. D., F. L. S., &c. &c. &c. Small 8vo, with Eight Illustrations. 5s. cloth.

CAMPBELL.—SPEECHES OF LORD CAMPBELL,
At the Bar and in the House of Commons; with an Address to the Irish Bar as Lord Chancellor of Ireland. 8vo, 8s. cloth.

CAMPBELL.—A MEMOIR ON EXTRA UTERINE GESTATION.
By Dr Wm. Campbell of Queen's College, Edinburgh, F. R. C. S., &c. &c. 8vo, 6s. 6d. boards.

CANADA.—VIEWS OF CANADA AND THE COLONISTS;
Embracing the Experience of a Residence; Views of the Present State, Progress, and Prospects of the Colony; with detailed Practical Information for intending Emigrants. By a Four Years' Resident. Fcap. 8vo, with a Map, 4s. 6d. cloth.

"The work of a shrewd and truthful observer."—Colonial Gazette.

CARRUTHERS.—THE HIGHLAND NOTE-BOOK;
Or Sketches and Anecdotes. By R. Carruthers, Inverness. 12mo, 4s. cloth.

"The present Volume is more than usually interesting, as giving us glimpses into Macbeth's country, the mountain and moorland fastnesses, in which the young Chevalier's Rebellion was closed, and the District swept by the Morayshire Floods, made classical ground by the delightful narrative of Sir Thomas Dick Lauder."—Athenæum.

CARSON.—PHÆDRI FABULÆ,
Augusti Liberti Fabularum Aesopiarum, quas oculis puerorum subjici fas est, libras quinque, cum Indice verborum phrasiumque difficiliorum Anglice redditorum. Edidit A. R. Carson, LL. D., Late Rector of the High School, Edinburgh. Editio Sexta. 18mo, 2s. bound.

CHRISTISON.—A DISPENSATORY;
Being a Commentary on the Pharmacopœias of Great Britain, comprising the Natural History, Description, Chemistry, Pharmacy, Actions, Uses, and Doses of the Articles of the Materia Medica. By Robert Christison, M. D., Professor of Materia Medica in the University of Edinburgh. New and improved Edition, with an Appendix on New Medicines. 8vo, 18s. cloth.

"We earnestly recommend Dr Christison's Dispensatory to all our readers, as an indispensable companion, not in the Study only, but in the 'Surgery' also."—Brit. and For. Med. Review.

CHRISTISON.—A TREATISE ON POISONS,
In relation to Medical Jurisprudence, Physiology, and the Practice of Physic. By Robert Christison, M. D., Professor of Materia Medica in the University of Edinburgh, &c. &c. Fourth Edition, enlarged, corrected, and improved. 8vo, 20s. cloth.

"It is beyond comparison the most valuable Practical Treatise on Toxicology extant."—London Medical and Physical Journal.

CHRISTISON.—THE KIDNEYS,
Being a Treatise on Granular Degeneration of the Kidneys, and its connection with Dropsy, Inflammation, and other Diseases. By Robert Christison, M. D., Professor of Materia Medica in the University of Edinburgh. 8vo, 8s. cloth.

"The illustrative cases, thirty-one in number, are narrated with Dr Christison's usual clearness, and, like the rest of the work, are highly instructive. We strongly recommend this book to our readers."—London Medical Gazette.

CLEGHORN.—A SYSTEM OF AGRICULTURE.
By James Cleghorn, Esq. With Thirteen Engravings. 4to, 9s. cloth.

"The best account of the Agriculture of the Scotch Counties is to be found in Black's Edition of the Encyclopædia Britannica."—Loudon's Agriculture, p. 1178.

CLERK.—NAVAL TACTICS,
Being a Systematical and Historical Essay, in Four Parts. By John Clerk, Esq. of Eldin, F. R. S. E., &c. Third Edition, with Notes by Lord Rodney, an Introduction by a Naval Officer, and explanatory Plates. 8vo, 25s. cloth.

COGSWELL.—IODINE.
An Experimental Essay on the relative Physiological and Medicinal properties of Iodine and its Compounds; being the Harveian Prize Dissertation for 1837. By Charles Cogswell, A. B., M. D., Member of the Royal College of Surgeons, Edinburgh, &c. &c. 8vo, 5s. boards.

COOPER.—A DICTIONARY OF PRACTICAL SURGERY:
Comprehending all the most interesting improvements from the earliest times down to the present Period; an account of the Instruments and Remedies employed in Surgery; the etymology and signification of the principal Terms; and numerous References to ancient and modern works, forming a catalogue of Surgical Literature, arranged according to subjects. The Seventh Edition, revised, corrected, and enlarged. By Samuel Cooper, Senior Surgeon to the University College Hospital, London, Professor of Surgery in the same College, Surgeon to the Queen's Bench, &c. &c. &c. Thick 8vo, 30s. cloth.

COOPER.—FIRST LINES OF SURGERY,
Being a Treatise on the Theory and Practice of Surgery, including the principal operations. By Samuel Cooper, Senior Surgeon to the University College Hospital, London, and Professor of Surgery in the same College. Seventh Edition. 8vo, 18s. cloth.

CRAIGIE.—ELEMENTS OF THE PRACTICE OF PHYSIC,
Presenting a View of the present state of Special Pathology and Therapeutics. By David Craigie, M. D., F. R. S. E., Fellow of the Royal College of Physicians, Edinburgh, Physician to the Royal Infirmary, Emeritus President and Extraordinary Member of the Royal Medical Society, &c. &c. &c. 2 volumes 8vo, 40s. cloth.

"We are inclined to regard Dr Craigie's Elements as the best we at present possess."—London Medical Gazette.

CRAIGIE.—ELEMENTS OF ANATOMY,
General, Special, and Comparative. By David Craigie, M. D., forming the article under that head in the Seventh Edition of the Encyclopædia Britannica. With Fourteen highly finished Engravings. 4to, 12s. cloth.

"Without branching out into unnecessary details, the leading points of each division are placed before the reader in a clear and concise (though sufficiently comprehensive) manner, the whole forming a volume which may be perused with pleasure and advantage, both by the non-professional man of science and the practical anatomist."—Lancet.

CREUZE.—SHIPBUILDING.
Being a Treatise on the Theory and Practice of Naval Architecture. By Augustin F. B. Creuze, Member of the late School of Naval Architecture; President of the Portsmouth Philosophical Society; and Editor of the "Papers on Naval Architecture." With 15 Engravings on Steel, and numerous Woodcuts. 4to, 12s. cloth.

"One of the best, because the clearest, and, at the same time, perfectly comprehensive disquisitions on Shipbuilding, is contained in the Encyclopædia Britannica."—Liverpool Mail.

CULLEN.—FIRST LINES OF THE PRACTICE OF PHYSIC.
By William Cullen, M. D., formerly Professor of Physic in the University of Edinburgh, &c. &c. &c. A new Edition, with an Appendix, by James Craufurd Gregory, M. D., F. R. C. S., &c. &c. 2 vols. 8vo, 24s. cloth.

CUNNINGHAM.—THE GEOLOGY OF THE LOTHIANS:
A Prize Essay, with 35 Coloured Sections, and a Geological Map of the Lothians. By R. J. H. Cunningham, Esq. M. W. S., &c. 8vo, 18s. cloth. This volume forms the Seventh volume of the Memoirs of the Wernerian Natural History Society, and includes Dr Parnell's Prize Essay on the Fishes of the District of the Forth, with 67 illustrative Figures; and the History of the Society from December 1831 till April 1838.

DICK.—A MANUAL OF VETERINARY SCIENCE;
Forming the Article under that head in the Seventh Edition of the Encyclopædia Britannica. By William Dick, Professor of Veterinary Surgery to the Highland Society of Scotland. Post 8vo, 3s. boards.

"All Farmers, and Cattle-dealers, Shepherds, Stablers, Coach Contractors, every man who is interested in the study of Veterinary Medicine, should have Mr Dick's Manual in his possession."—Edinburgh Advertiser.

DUTRUC.—A FRENCH GRAMMAR,
On a New and Original Plan, exhibiting the Pronunciation of the French in pure English Sounds; with Conversazione Exercises, preceded by a Method of Questioning and Answering; the whole particularly adapted to smooth the way to a perfect acquaintance with the French Language. To which is subjoined an extensive English and French Mercantile Correspondence, chiefly collected from the best writers on the subject. 8vo, 6s. cloth.

EDINBURGH MEDICAL AND SURGICAL JOURNAL,
Exhibiting a concise View of the latest and most important Discoveries in Medicine, Surgery, and Pharmacy. Publishing in Quarterly Numbers, 6s. sewed.

In every branch of Medical and Surgical Science, this work exhibits a valuable compendious view of the progressive advancement of knowledge; and to those who desire to keep pace with the improvements of the times, either in the theoretical departments of Medical Science, or in their practical application, the Edinburgh Medical and Surgical Journal is recommended, not only as a safe and judicious guide, but as an indispensable requisite.

EDINBURGH NEW PHILOSOPHICAL JOURNAL,
Exhibiting a View of the Progressive Discoveries and Improvements in the Sciences and the Arts. Conducted by Professor Jameson. Publishing in Quarterly Numbers, 7s. 6d.

The Edinburgh New Philosophical Journal is now the only Journal of Science published out of London. Its plan embraces all the subjects regarded as most important by those engaged in scientific pursuits; and for the benefit of the general reader, the information is conveyed in a form as popular as the nature of the subjects will admit of. The numbers will be found to contain many important articles illustrating the progress of discovery in Natural Philosophy, Geology, Chemistry, Natural History, Comparative Anatomy, Practical Mechanics, Geography, Navigation, Statistics, Antiquities, and the Fine and Useful Arts; together with many interesting Biographies of Individuals eminent in the history of Science.

EDINBURGH PHARMACOPŒIA OF THE ROYAL COLLEGE OF PHYSICIANS. Second English Edition. 18mo, 5s. cloth.

ENCYCLOPÆDIA BRITANNICA.
The Seventh Edition, edited by Professor Napier. Illustrated by 506 Engravings on Steel, and many Thousands on Wood, with a general Index of upwards of 68,000 References. Twenty-one volumes Quarto, L.37, 10s. cloth, L.42 half bound russia or morocco.

"This is the really cheap Encyclopædia, for that only is cheap which is excellent."* "It is our firm conviction, that the British public was never before in possession of a work of this class so complete in all its departments, so varied in its subjects, so profound in its information, or sanctioned with the stamp of such great authorities in every branch of the Arts, Sciences, and General Literature."† "The first thinkers, writers, and savans of the age, have in this book consolidated a body of Philosophy, History, Belles Lettres, Biography, Science, and Art, which fulfils whatever is required from a complete circle of Literature and the Sciences."‡ "With the exception of such articles as remain comparatively unaffected by the progress of time and improvement, by far the greater number have been written expressly for this Edition, and, in every instance, by persons the most eminent in the respective departments of knowledge of which they have been employed to treat."§ "To such an extent has this substitution of new matter been carried, that in some of the volumes, three-fourths of the whole contents are entirely new."‖ "For extent of usefulness, and moderation of price, it appears to us unparalleled; and to men whose fortunes and rooms do not permit them the enjoyment of a large collection of books, we cannot conceive a more valuable acquisition than this edition of that most comprehensive of all works—the Encyclopædia Britannica."¶

"The Index, a work of stupendous labour, and, without doubt, the most comprehensive index of miscellaneous literature and science in the English language, exhibits in a very striking manner the vast extent of information contained in the Encyclopædia, and must prove of invaluable service to all who have occasion to consult books of reference."**

"The quality of the paper,—the remarkable elegance of the typography,—and the beauty of the plates, combined with the literary value of its contents, render it the best, and because the best,—the cheapest Encyclopædia ever published in Britain."†† "Now that the country is being deluged with diluted stuff, compounded from Germany and America, what hope is there for the Science and the Literature of England, that publishers dare ever again venture on such another work as this?"‡‡

"To the Gentleman and the Merchant, to the Agriculturist and the Manufacturer, to the Clergyman and the Layman, to the Student of Science or Philosophy, and the Cultivator of Literature or the Fine Arts, the ENCYCLOPÆDIA BRITANNICA will prove an acquisition of the highest value. The great scope of its information also recommends it in an especial manner to emigrants, and other persons resident in quarters where access to books is difficult, or whose fortunes do not permit them the enjoyment of extensive libraries."§§ "An Australian or New Zealand settler, who left his home with no other accomplishment but that of being able to read, write, and count, might, with such a companion, beguile his long and weary voyage, and become a well-informed man before he reached his destination."‖‖ As a token of regard presented by individuals or associations, no gift could be devised better calculated to fulfil the objects of such testimonials; for while the beauty and splendour of the work commend it to the taste of the donor, its great practical utility cannot fail to render it highly acceptable to the party receiving it.

* Athenæum.	† Dublin Evening Mail.	‡ Tait's Magazine.
§ Morning Chronicle.	‖ Bradford Observer.	¶ John Bull.
** Caledonian Mercury.	†† Bristol Journal.	‡‡ Athenæum.
§§ Leeds Conservative Journal.		‖‖ Quarterly Review.

FLEMING.—MOLLUSCOUS ANIMALS,
Including Shell Fish; containing an Exposition of their Structure, Systematic Arrangement, Physical Distribution, and Dietetical Uses, with a reference to the Extinct Races. By John Fleming, D.D., F.R.S.E., M.W.S., Professor of Natural Philosophy in the University and King's College, Aberdeen, &c. &c. &c. With Eighteen Plates. Post 8vo, 6s. cloth.

"Distinguished by a perfect knowledge of the very curious and interesting subject of which it treats, by a severe and searching analysis of the evidence, and a clear and masterly arrangement of the multifarious details connected with it."—GLASGOW CONSTITUTIONAL.

FORBES.—TRAVELS THROUGH THE ALPS OF SAVOY,
And other Parts of the Pennine Chain, with Observations on the Phenomena of Glaciers. By James D. Forbes, F.R.S., Sec. R.S.E., Corresponding Member of the Royal Institute of France, Professor of Natural Philosophy in the University of Edinburgh, &c. &c. A New Edition, Revised and Enlarged. Illustrated by a large Map of the Mer de Glace of Chamouni, Lithographed Views and Plans, and Engravings on Wood. Imperial octavo, 28s., or with the large Map coloured, in a Case, 31s. 6d. cloth.

"This elaborate and beautifully illustrated work."—QUARTERLY REVIEW.
"Pregnant with interest."—EDINBURGH REVIEW.

FYFE.—THE ANATOMY OF THE BODY;
Illustrated by One Hundred and Fifty-Eight Plates taken partly from the most celebrated Authors, partly from Nature. By Andrew Fyfe, F.R.S.E. 4to, boards, with Descriptive letterpress, 8vo, 40s. boards.

GALLOWAY.—A TREATISE ON PROBABILITY.
By Thomas Galloway, M.A., F.R.S., Secretary to the Royal Astronomical Society. Post 8vo, 6s. cloth.

GIBBON—DECLINE AND FALL OF THE ROMAN EMPIRE.
By Edward Gibbon, Esq. New Edition, in eight volumes, 8vo, 63s. cloth.

GIBBON.—DECLINE AND FALL OF THE ROMAN EMPIRE.
By Edward Gibbon, Esq. A new Edition, with some account of the Life and Writings of the Author, by Alexander Chalmers, Esq. F. S. A., and a Portrait. In one thick volume, 8vo, 20s. cloth.

GIBSON.—EUTROPII HISTORIÆ ROMANÆ BREVIARII
Primas Quinque Libros; Notis Anglicis, et Vocabulario illustravit Gilbertus M. Gibson, in Academia apud Oppidum Bathgate Rector. 18mo, 2s. 6d. cloth.

GIBSON.—LE PETIT FABLIER
Ou Esope en Miniature. Par T. G. Gibson, Maitre de Cauvin's Hospital, &c. &c. 12mo, 1s. 6d. cloth.

GILLY.—VALDENSES VALDO, AND VIGILANTIUS.
By the Rev. W. S. Gilly, D. D., Author of "Waldensian Researches." Post 8vo, 2s. 6d.

"An eloquent account, from personal observation, of that small community of Protestants, who, in the secluded valleys of the Cottian Alps, have, for many centuries, maintained the purity of their faith and worship, and kept up the fire of their vestal church, in the midst of privations and persecutions not yet extinguished."—Quarterly Review.

GLASSFORD.—ITALIAN POETS.
Being Lyrical Compositions selected from the Italian Poets, with Translations. By James Glassford, Esq. of Dougalston. Second Edition, greatly enlarged. Small 8vo, cloth.

GLASSFORD.—SPHINX INCRUENTA,
Or Two Hundred and Twelve original Enigmas and Charades. By James Glassford, Esq. of Dougalston. 12mo, 3s. 6d. cloth.

GRAHAM.—MUSICAL COMPOSITION.
Being an Essay on its Theory and Practice, with an Appendix and Notes, being an extension of the article "Music." By G. F. Graham, Esq. With numerous Engravings, and copious Musical Illustrations interspersed with the Text. 4to, 9s. boards.

"A masterly and comprehensive Essay."—Athenæum.

GRAY.—THE SOCIAL SYSTEM.
A Treatise on the Principles of Exchange. By John Gray. 8vo, 7s. cloth.

GRAY.—REMEDY FOR THE DISTRESS OF NATIONS.
By John Gray. 8vo, 6s. cloth.

HANSARD.—PRINTING AND TYPE-FOUNDING.
Two Treatises by T. C. Hansard. Illustrated with Plates and Woodcuts. Post 8vo, 6s. cloth.

"A Printer's Manual which every one in the trade will find it his interest to possess."—Westminster Review.

HAYDON AND HAZLITT.—PAINTING AND THE FINE ARTS.
By B. R. Haydon and William Hazlitt. Post 8vo, 6s. cloth.

"Mr Hazlitt's clever little Treatise, written for the 'Encyclopædia Britannica,' has come under our notice. We have read no work of that author with anything approaching to the same gratification."—Quarterly Review.

HENDERSON.—ASTRONOMICAL OBSERVATIONS
Made at the Royal Observatory, Edinburgh. By Thomas Henderson, F. R. S. E., and R. A. S., Professor of Practical Astronomy in the University of Edinburgh, and Her Majesty's Astronomer for Scotland. Published by order of Her Majesty's Government. Vols. 1, 2, 3, 4, and 5. Quarto, 12s. boards.

HETHERINGTON.—THE HISTORY OF ROME.
By the Rev. W. M. Hetherington, M. A. With an Account of the Topography and Statistics of Modern Rome. By the Rev. J. Taylor, M. A.

*** Without suppressing those traditionary legends which are blended with the earlier records of Roman History, an attempt has been made in the present work to distinguish between Fact and Fable, and to make the latter subserve the important purpose of elucidating and of fixing in the Memory those real events from which the fabulous legends have arisen. With a Map of Ancient Rome. 6s. bound.

HINDMARSH.—A SELECTION OF SACRED EPIC POETRY;
From Milton, Montgomery, Cumberland, Young, Porteous, and Blair. (Marked with Emphases, Inflections, and Rhetorical Pauses.) Intended, in connection with a proportion of Scriptural Reading, as a regular Sunday Lesson Book. By J. H. Hindmarsh, Teacher of Elocution, Perth Academy. 18mo, 1s. 6d. bound.

HOOPER.—LEXICON MEDICUM;
Or Medical Dictionary: containing an explanation of the terms in Anatomy, Human and Comparative, Botany, Chemistry, Materia Medica, Midwifery, Pharmacy, Physiology, Practice of Physic, Surgery, and the various branches of Natural Philosophy connected with Medicine. Selected, arranged, and compiled from the best Authors. By the late Robert Hooper, M.D., F. L. S. The Seventh Edition, revised, corrected, and enlarged. By Klein Grant, M. D., &c. &c., Lecturer on Therapeutics at the North London School of Medicine. Thick 8vo, 30s. cloth.

MALTE-BRUN.—PHYSICAL GEOGRAPHY.
Principles of Mathematical, Physical, and Political Geography. By M. Malte-Brun. Second Edition. Improved by the addition of the most recent information, derived from various sources. With Four Plates. 8vo, 15s. cloth.

"We think the translators of Malte-Brun's Geography, have done good service to the public, by rendering so valuable a work accessible to the English reader."—Edinburgh Review.

MALTE-BRUN.—UNIVERSAL GEOGRAPHY.
Or a Description of all the Parts of the World. By M. Malte-Brun, Editor of the "Annales des Voyages," &c. &c. With a Comprehensive Index of 44,000 Names. Nine volumes, 8vo, L.7, cloth.

MARSHALL.—ON SOLDIERS.
The Enlisting, Discharging, and Pensioning of Soldiers, with the Official Documents on these Branches of Military Duty. By Henry Marshall, F. R. S. E., Deputy-Inspector-General of Army Hospitals. Second Edition. 8vo, 7s. 6d. cloth.

"A most valuable book, and ought to be in the library of every medical officer in the public service, whether of the army or navy."—British and Foreign Medical Review.

MARTIN.—THE SEAMAN'S ARITHMETIC;
Or the Application of the several Rules of that Science to the purposes of the Maritime Profession; being intended as an Introduction to the Young Mariner preparatory to his learning Navigation, and for his subsequent guidance. By John Martin, Agent, Kincardine. 18mo, 2s. 6d. cloth.

"We doubt not 'The Seaman's Arithmetic' will meet with that general favour to which its intrinsic excellence and useful capabilities so fully entitle it."—Caledonian Mercury.

MARTIN, (AIME.)—THE EDUCATION OF MOTHERS OF FAMILIES;
Or the Civilization of the Human Race of Women. By M. Aime Martin. Being the work to which the Prize of the French Academy was awarded. Translated from the Third Paris Edition. By Edwin Lee, Esq., Member of the Principal European Medical and Chirurgical Societies, Author of the Baths of Germany, &c. &c. 8vo, 10s. 6d. cloth.

MATHER.—AN ACCOUNT OF THE SHIP LIFE BOAT.
By James Mather, Esq. Member of the Royal Physical Society, Member of the Wernerian Society, &c. &c. Second Edition. 8vo, 1s. sewed.

MATTHEW.—EMIGRATION FIELDS:
North America, the Cape, Australia, and New Zealand, describing these Countries, and giving a comparative view of the advantages they present to British Settlers. By Patrick Matthew, Author of "Naval Timber and Arboriculture." With two Folio Maps, engraved by Sydney Hall. Post 8vo, 3s. 6d. cloth.

"The information contained in this work is of such a nature, that every one who has an intention of emigrating, should, before fixing upon any country as his future residence, consult the Emigration Fields."—Dundee Chronicle.

MATTHEW.—NAVAL TIMBER AND ARBORICULTURE.
Being a Treatise on that subject, with Critical Notes on Authors who have recently treated the subject of planting. By Patrick Matthew. 8vo, 12s. cloth.

MENZEL.—EUROPE IN 1840.
Translated from the German of Wolfgang Menzel. Post 12mo, 5s. cloth.

"A work which every politician will do well to consult."—Eclectic Review.

MILLER.—THE PRINCIPLES OF SURGERY.
By James Miller, F. R. S. E., Professor of Surgery in the University of Edinburgh, &c. &c. Small 8vo, 9s. cloth.

"An admirable epitome of the surgical science of the day. Being written by a sound practical surgeon, accustomed to the public teaching of his science, it has that clearness of diction and arrangement which renders it an excellent manual for the student, as well as that amount of scientific and practical information which makes it a safe and valuable guide to the practitioner."—Lancet.

MILLER.—THE PRACTICE OF SURGERY.
By James Miller, F. R. S. E., Professor of Surgery in the University of Edinburgh, &c. Uniform with "the Principles."

"We have no hesitation in stating that the two volumes form, together, a more complete textbook of surgery than any one that has been heretofore offered to the student."—Northern Journal of Medicine.

MILLER.—SCENES AND LEGENDS OF THE NORTH OF SCOTLAND.
By Hugh Miller. Small 8vo, 7s. 6d. cloth.

"A well imagined, a well written, and a somewhat remarkable book."—Athenæum.

"A very pleasing and interesting book; his style has a purity and elegance which remind one of Irving, or of Irving's master, Goldsmith."—Spectator.

"A highly amusing and interesting book, written by a remarkable man, who will infallibly be well known."—Leigh Hunt's Journal.

MILLER.—PICTORIAL ANATOMY.
An Introductory Lecture delivered to the Students of the School of Design. By James Miller, F. R. S. E., &c. &c., Professor of Surgery in the University of Edinburgh. 8vo, 1s. sewed.

MOIR AND SPALDING.—POETRY, ROMANCE, AND RHETORIC.
By William Spalding, Professor of Rhetoric in the University of Edinburgh, and George Moir, Esq. Advocate. Post 8vo, 6s. cloth.

"The article 'Poetry, by Professor Moir,' is prefaced by an eloquent and perspicuous exposition of the mental and moral qualities requisite to its production; and the distinction between the province of imagination and of fancy evinces discrimination at once accurate and profound."—Exeter Gazette.

MOORE.—ELEMENTA LINGUÆ GRÆCÆ.
Studio Jacobi Moore, LL. D. Emendavit auxitque Jacobus Tate, A. M., Cantab. Editio Quinta Correctior.
To this Edition is added an Appendix, containing a list of Irregular and Defective Greek Verbs, and a Synopsis of the Indeclinable Parts of Speech. 12mo, 2s. 6d. bound.

MUSHET.—WRONGS OF THE ANIMAL WORLD.
To which is subjoined the Speech of Lord Erskine on the same subject. By David Mushet, Esq. 8vo, 8s. cloth.

NEILL.—THE FRUIT, FLOWER, AND KITCHEN GARDEN.
By Patrick Neill, LL. D., F. R. S. E., Secretary to the Caledonian Horticultural Society. Third Edition. Revised and improved, illustrated with upwards of 60 Woodcuts. Post 8vo, 6s. cloth.

"One of the best modern books on Gardening extant."—Loudon's Gardener's Magazine.

"Practical gardeners and amateurs owe a debt of gratitude to him for his excellent work on Horticulture, which is now one of the standard works on the branch of science of which it treats."—Professor Dunbar's Speech in the Caledonian Horticultural Society.

NIMROD.—THE HORSE AND THE HOUND;
Their various Uses and Treatment, In-doors and Out; including Practical Instructions in Horsemanship. To which is added, a Treatise on Horse-Dealing, wherein is enforced the necessity for "Caveat emptor," and a recital given of some of the first Legal and Veterinary Authorities on the question of Soundness and Unsoundness of Horses. By Nimrod. Second Edition, with numerous Illustrations on Wood and Steel, after Drawings by Cooper, Alken, Barenger, and Ferneley of Melton Mowbray. Post 8vo, 12s. cloth.

NUGENT.—POCKET DICTIONARY OF THE FRENCH AND ENGLISH LANGUAGES. In two Parts, 1st, French and English; 2d, English and French, containing all Words of General Use, and authorised by the best Writers. As also the several Parts of Speech. The Genders of the French Nouns, the Naval and Military Terms, a List of Proper Names, &c. &c. A new Edition, carefully revised and enlarged by J. Ouiseaux, A. M. Square 12mo, 7s. 6d. bound; or Pearl, 5s. 6d. bound.

OSWALD.—AN ETYMOLOGICAL DICTIONARY OF THE ENGLISH LANGUAGE, on a Plan entirely new, adapted to the Modern System of Tuition. By the Rev. John Oswald, late Master in George Heriot's Hospital. Fourth Edition. 18mo, 8s. 6d. bound.

"This very laborious compilation deserves to be introduced into every school where the English Language is taught."—Athenæum.

OSWALD.—AN ETYMOLOGICAL MANUAL OF THE ENGLISH LANGUAGE, for the Use of Schools and Private Families. Eleventh Edition, price 1s. 6d. cloth.

OSWALD.—OUTLINES OF ENGLISH GRAMMAR.
Fifth Edition, price 6d.

OSWALD.—ETYMOLOGICAL PRIMER.
Part First. Twelfth Edition, 1d.

OSWALD.—ETYMOLOGICAL PRIMER.
Part Second. Fourth Edition, 6d.

PARNELL.—FISHES OF THE FRITH OF FORTH;
Their Natural and Economical History. By Richard Parnell, M. D., F. R. S. E. With 67 Figures. 8vo, 8s. cloth.

PEEL.—SIR ROBERT PEEL AND HIS ERA.
Post 8vo, 7s. 6d. cloth.

PHILLIPS.—A TREATISE ON GEOLOGY.
By John Phillips, F. R. S., F. G. S., Professor of Geology in King's College, London, Author of "Illustrations of the Geology of Yorkshire," "A Guide to Geology," &c. &c. Embellished with Plates and Woodcuts. Post 8vo, 6s. cloth.

"The author has selected and combined all the discoveries which have been made in Geology up to the present time."—Morning Herald.

PILLANS.—PRINCIPLES OF ELEMENTARY TEACHING,
Chiefly in reference to the Parochial Schools of Scotland: in Two Letters to T. F. Kennedy, Esq. M. P. By James Pillans, F. R. S. E., late Rector of the High School, and now Professor of Humanity in the University of Edinburgh. Post 8vo, 3s. boards.

PILLANS.—ECLOGÆ CICERONIANÆ.
A Selection from the Orations, Epistles, and Philosophical Dialogues of Cicero: to which are added selected Letters of Pliny the Younger. Arranged in three distinct divisions; being intended to give the studious youth a specimen of Cicero's compositions in different aspects of his Character: 1. as an Orator; 2. as a Man of the World and a Member of Society; and 3. as a Philosopher. By James Pillans, F. R. S. E., Professor of Humanity in the University of Edinburgh. 12mo, 3s. 6d. cloth.

PILLANS.—ON CLASSICAL EDUCATION.
The Proper Objects and Methods of Education, in reference to the different Orders of Society, and on the relative utility of Classical Instruction: being Three Lectures delivered in the University of Edinburgh. By James Pillans, M. A., F. R. S. E., Professor of Humanity in that University. 8vo, 2s. sewed.

RAMSAY.—AN ESSAY ON THE DISTRIBUTION OF WEALTH.
By George Ramsay, B. M. of Trinity College, Cambridge. 8vo, 12s. cloth.

RAMSAY.—A DISQUISITION ON GOVERNMENT.
By George Ramsay, B. M. of Trinity College, Cambridge. Foolscap, 4s. cloth.

RAMSAY.—POLITICAL DISCOURSES.
1. On what is Government founded; 2. On Civil Liberty; 3. On Vote by Ballot; 4. On Equality and Inequality; 5. On Central and Local Systems. By George Ramsay, B. M. Trinity College, Cambridge. 8vo, 9s. cloth.

ROBERTSON.—REPORT OF THE AUCHTERARDER CASE.
The Earl of Kinnoull and the Rev. R. Young against the Presbytery of Auchterarder. By Charles Robertson, Esq. Advocate, one of the Collectors of Decisions by appointment of the Faculty of Advocates. Published by authority of the Court. Two volumes, with Supplement. 8vo, 20s. 6d. boards.

ROBERTSON.—COLLOQUIA DE MORBIS.
Practica et Theoretica, Questionibus et Responsis; ad usum Ingenuae Juventutis accommodata. Auctore Archibaldo Robertson, M. D., &c. &c. Editio Tertia. 18mo, 7s. 6d. boards.

ROD, THE, AND THE GUN.
Being Two Treatises on Angling and Shooting. The former by James Wilson, Esq. F. R. S. F., &c. &c. The latter by the Author of "The Oakleigh Shooting Code." Second Edition, with numerous Engravings on Wood and Steel. Post 8vo, 10s. 6d. cloth.

"The treatise on Shooting, by the Author of 'The Oakleigh Shooting Code,' is written upon a very comprehensive plan, and beautifully illustrated. Its companion on Angling is one of the most interesting, instructive, and agreeable treatises on 'the gentle art' that exists in our language; and will probably be noticed at greater length in a future article."—Edinburgh Review.

"Know likewise to thy utter discomfort, nay, to thy utter confusion, that a book has lately appeared yclept 'The Rod and the Gun,' so amusingly written and so complete in all its parts, that there is not the least occasion for you to burthen Mr Murray's shelves with stale precepts that no one will attend to."—Preface to "Days and Nights of Salmon Fishing" by William Scrope, Esq.

ROGET.—PHYSIOLOGY AND PHRENOLOGY.
By P. M. Roget, M. D., Secretary to the Royal Society, &c. &c., Author of the Fifth Bridgewater Treatise. Two volumes, post 8vo, 12s. cloth.

"A luminous and most candid and impartial account of Phrenology. . . . In the Treatise on Physiology, that science is treated clearly, fully, and in the systematic manner which a masterly instructor might adopt for the benefit of his pupils."—Tait's Magazine.

ROLLIN.—ANCIENT HISTORY
Of the Egyptians, Carthaginians, Assyrians, Babylonians, Medes and Persians, Macedonians, and Grecians. By M. Rollin, late Principal of the University of Paris, &c. &c. Translated from the French. New Edition. Illustrated with Maps and other Engravings. In Six Volumes. 8vo, 42s. cloth.

RUSSELL.—THE HISTORY OF MODERN EUROPE:
With an Account of the Decline and Fall of the Roman Empire, and a View of the Progress of Society from the Rise of the Modern Kingdoms to the Peace of Paris in 1763; in a Series of Letters from a Nobleman to his Son. New Edition, continued to the Accession of Queen Victoria of England. Four Volumes, 8vo, 52s. cloth.

RUSSELL.—A TREATISE ON THE STEAM ENGINE.
Illustrated by 248 Engravings on Wood, and 15 Folding Plates on Steel. Post 8vo, 9s. cloth.

"Most complete and circumstantial. . . . . At the same time it is methodically, clearly, and luminously written. Considering the number of illustrations, it is a very cheap book, and as it explains all the modern improvements and applications, it cannot fail in being a boon which every mechanist and engineer will receive with much gratitude."—The Surveyor, Engineer, and Architect.

RUSSELL.—STEAM AND STEAM NAVIGATION.
A Treatise on the Nature, Properties, and Applications of Steam, and on Steam Navigation. By John Scott Russell, M. A., F. R. S. E., Vice-President of the Society of Arts of Scotland. Illustrated with upwards of 80 Engravings on Wood, and 15 Folding Plates on Steel. Post 8vo, 9s. cloth.

"A work on Steam and Steam Navigation, in which science and interesting information are equally combined."—Mechanics' Magazine.

SHAW.—DEVELOPMENT AND GROWTH OF SALMON FRY,
Being Experimental Observations from the Exclusion of the Ova to the Age of Two Years. By John Shaw, Drumlanrig. With two Engravings, 4to, 2s. 6d. sewed.

"That these facts may not be regarded as the result of hasty or of superficial observation, we may mention that the ingenious author has resided almost during his entire life by the banks of salmon streams, and that his opportunities have thus been as ample as we know his efforts have been unremitting and laborious, to ascertain the genuine history of this noble and most valuable species."—BLACKWOOD'S MAGAZINE, APRIL 1840.

SHOOTER'S HAND-BOOK,
Being the Treatise on Shooting contained in "The Rod and the Gun." By the Author of "The Oakleigh Shooting Code." With Plates and Woodcuts. Post 8vo, 6s. cloth.

*** The "Oakleigh Shooting Code" is now withdrawn from circulation, the present volume embracing all the contents of that work which the author deemed worthy of preservation, as well as much new matter, the result of his enlarged experience and more matured opinions.

SHORTREDE.—LOGARITHMIC TABLES TO SEVEN PLACES OF DECIMALS, containing Logarithms to Numbers from 1 to 120·000, Numbers to Logarithms from .0 to 1·00000, Logarithmic Sines and Tangents to every second of the Circle, with Arguments in Space and Time, and New Astronomical and Geodesical Tables. By Robert Shortrede, F. R. A. S., &c., Captain H. E. I. C. S., and First Assistant of the Great Trigonometrical Survey of India. Imperial 8vo, 42s. cloth.

SIMPSON.—THE PHILOSOPHY OF EDUCATION,
With its Practical Application to a System and Plan of Popular Education as a National Object. By James Simpson, Esq. Advocate. Second Edition. 12mo, 4s. boards.

"It should be in the hands of every friend of his species who can afford to purchase it."—LEIGH HUNT'S JOURNAL.

SMITH.—THE WEALTH OF NATIONS.
An Inquiry into the Nature and Causes of the Wealth of Nations. By Adam Smith, LL. D.; with the Life of the Author, an Introductory Discourse, Notes, and Supplemental Dissertations, by J. R. M'Culloch, Esq. A new Edition, corrected throughout, and greatly enlarged. With two Portraits. 8vo, 21s. cloth.

*** This Edition contains elaborate Notes on the Corn Laws, the Poor Law Act, the Colonies, &c.

STARK.—PICTURE OF EDINBURGH,
Containing a Description of the City and its environs. By J. Stark, F. R. S. E. Sixth Edition, improved. With a new Plan of the City, and Forty-Eight Views of the Principal Buildings. 18mo, 8s. roan.

STEVENSON.—MARINE SURVEYING AND HYDROMETRY.
Being a Treatise on their Application to the Practice of Civil Engineering. By David Stevenson, Civil Engineer, Author of "A Sketch of the Civil Engineering of America," &c. Illustrated by Thirteen Plates, a Coloured Chart, and numerous Engravings on Wood. Royal 8vo, 15s. cloth.

"A work of more extensive practical utility, more certain to bring honour to its author, and confer lasting benefit on his profession, has seldom come under our notice."—MECHANICS' MAGAZINE.

STRATTON.—GAELIC DERIVATIONS.
The Derivation of many Classical Proper Names from the Gaelic Language or the Celtic of Scotland: Being Part Third of an Inquiry into the partly Celtic Origin of the Greeks and Romans. By Thomas Stratton, M. D., Edinburgh, Honorary Member of the Highland Society, Canada. 12mo, 1s. 6d. sewed.

SYME.—DISEASES OF THE RECTUM.
A Treatise by James Syme, F. R. S. E., Professor of Clinical Surgery in the University of Edinburgh, &c. &c. 8vo, 5s. cloth.

THOMSON.—CHEMISTRY OF ANIMAL BODIES.
By Thomas Thomson, M. D., Regius Professor of Chemistry in the University of Glasgow, F. R. SS. L. and E., &c. &c. 8vo, 16s. cloth.

"The chymist who prefers his own investigation to the sometimes coloured views arrived at by others, will select the work we are now noticing as his text-book and guide to Animal Chymistry, in preference to all others which have appeared in this country."—ANNALS OF CHYMISTRY.

THOMSON.—THE LAW OF BILLS OF EXCHANGE,
Promissory Notes, &c. A Treatise by Robert Thomson, Esq. Advocate. Second Edition. Including a Summary of Decisions, Scotch and English, to the Present Time. 8vo, 24s. cloth.

TRAILL.—MEDICAL JURISPRUDENCE.
Being Outlines of a Course of Lectures by Thomas Stewart Traill, M. D., F. R. S. E., &c. &c., Regius Professor of Medical Jurisprudence and Medical Police in the University of Edinburgh. Second Edition. Post 8vo, 5s. cloth.

"These Outlines may indeed be recommended as presenting an excellent compendious view of the present state of the doctrines in Medical Jurisprudence; and they will serve as a useful guide, not only for initiating beginners, but for conducting the student in his subsequent prosecution of the subject, either generally or in its divisions."—EDINBURGH MEDICAL AND SURGICAL JOURNAL.

TRAILL.—PHYSICAL GEOGRAPHY.
By Thomas Stewart Traill, F. R. S. E., Regius Professor of Medical Jurisprudence in the University of Edinburgh, &c. &c. Post 8vo, 8s. cloth.

"A most elaborate digest of facts judiciously arranged, and, as a general exposition, perhaps the most complete that has yet appeared."—LEEDS MERCURY.